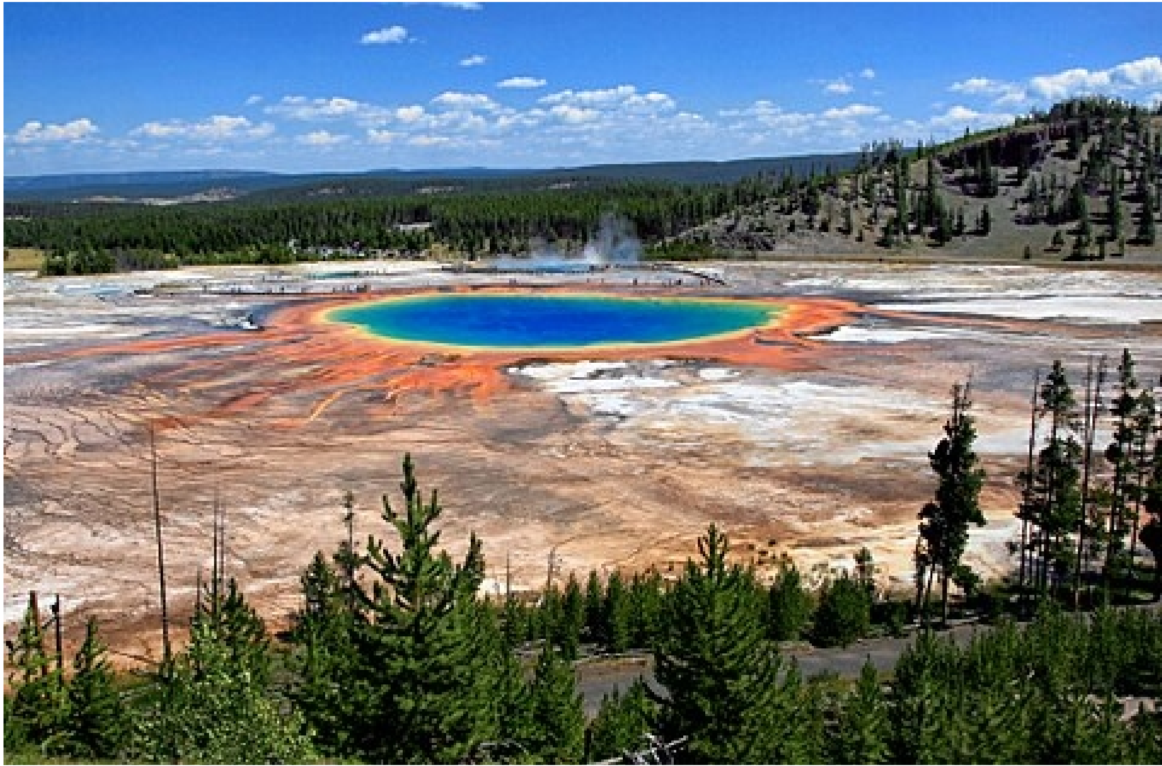
# The prismatic site

Grand Prismatic Spring, Yellowstone National Park



UCSD participants: please fill out the poll in the UCSD stream on Zulip.

# Presheaves on a topological ~~spaces~~ space as functors

$X$ = topological space $|X|$ = small category where objects = open subsets on $X$, morphisms are inclusions

A presheaf $\mathcal{F}$ on $X$ valued in some category $\mathcal{C}$ is a contravariant functor $\mathcal{F}: |X| \to \mathcal{C}$

A sheaf is a presheaf that preserves colimits of diagrams of form $\{V_i \cap V_j\} \rightrightarrows \{V_i\} \longrightarrow U$ where $U$ open covered by $\{V_i\}$.

# Indiscrete Grothendieck topologies

Let $\mathcal{C}$ be a (small) category. $Psh(\mathcal{C})$ = presheaves of abelian groups on $\mathcal{C}$

global sections

$F \mapsto H^0(\mathcal{C}, F) = \lim_{X \in \mathcal{C}} F(X)$

$\Gamma(\mathcal{C}, F)$

left exact $\Rightarrow$ right derived functors

$H^i(\mathcal{C}, F) = R\Gamma^i(\mathcal{C}, F)$

declare all presheaves are sheaves

# Example: inverse limits and R^1

$$\mathcal{C} = \{ 0 \to 1 \to 2 \to \cdots \}$$

$$H^0(\mathcal{C}, F) = \lim_{\substack{\leftarrow \\ n}} F(n) \quad H^1(\mathcal{C}, F) = R^1 \lim_{\substack{\leftarrow \\ n}} F(n)$$

and $\quad H^i(\mathcal{C}, F) = 0 \ \forall i \geq 2.$

# Weakly final objects

$\mathcal{C}$ = category

Final object is $X \in \mathcal{C}$ s.t. $\forall Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ is a singleton.

Weakly final object is $X \in \mathcal{C}$ s.t. $\forall Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(Y, X) \neq \emptyset$.

$K$ = field

e.g. $\mathcal{C} \subset \mathrm{Rings}_K$ full subcategory of ~~alg~~ algebraic field extensions

has weakly final objects (algebraic closures)

(related to Galois cohomology)

# Cohomology via a weakly final object

Lemma $\mathcal{C}$ = (small) category admitting finite nonempty products, suppose $X$ = weakly final.

Then for any $F \in \mathrm{Psh}(\mathcal{C})$, $R\Gamma(\mathcal{C}, F) \in D(Ab)$ equals the Čech-Alexander complex

$$0 \to F(X) \to F(X \times X) \to F(X \times X \times X) \to \cdots$$

derived from Čech nerve.

(Check equality to $H^0$ and check coeffaceable $\delta$-functor.)

# Preview: the Čech nerve as a simplicial object

$\Delta$ = category of finite ~~ordered~~ nonempty totally ordered sets (weakly ~~order~~ preserving maps))

$[0] \rightleftarrows [1] \rightleftarrows [2]$

$= \{0\} \quad \{0,1\} \quad \{0,1,2\}$

$\mathcal{C}$ = any category. A simplicial object is (contravariant) functor $\Delta^{op} \to \mathcal{C}$

cosimplicial object is cov. functor $\Delta \to \mathcal{C}$.

Given a cosimplicial object in Ab, set Č-A complex

$$d^n = \sum_{i=0}^{n} (-1)^i \delta_i^{n+1}$$

$$\begin{tikzcd}
X & X \times X & X \times X \times X \\
U([0]) \arrow[r, "\sigma^0_0"] & U([1]) \arrow[l, shift right=4, "\delta^1_1"'] \arrow[l, shift left=4, "\delta^1_0"] \arrow[r, shift right=2, "\sigma^1_1"'] \arrow[r, shift left=2, "\sigma^1_0"] & U([2]) \cdots \arrow[l, shift right=6, "\delta^2_2"'] \arrow[l, "\delta^2_1"] \arrow[l, shift left=6, "\delta^2_0"]
\end{tikzcd}$$

A generic simplicial object. For the Čech nerve of X, take U([n]) to be the (n+1)-fold product of X.

# The prismatic (oppo-)site

Let $(A, I) =$ prism

$\bar{A} = A/I$ slice

$R \in \mathrm{Ring}_{\bar{A}}$

naive prismatic opposite $(R/A)^{op}_{\Delta}$ is category of objects

$$\begin{tikzcd}
B/IB & B \arrow[l] \\
R \arrow[u] & \\
\bar{A} \arrow[u] & A \arrow[l] \arrow[uu]
\end{tikzcd}$$

$(A, I) \to (B, IB)$ +

$R \to B/I$

morphisms are morphisms of these diagrams (in Ring)

# Some sample objects of the prismatic site

If $R = \bar{A}$, then $\Delta_{R/A}$ final object. ($R = \bar{A} \leftarrow A$)

($B = A$)

but if $R \neq \bar{A}$, usually no final object

(but there is a weakly final object!)

$R = \bar{A}\langle X \rangle$ (completion of $\bar{A}[X]$) $\quad \delta(X) = 0$.

$B = (p, I)$-completion of $A[X]$

# Prismatic and Hodge-Tate cohomology

on $(R/A)_{\Delta}$ define (pre) sheaves of rings:

$\mathcal{O}_{\Delta} : (R \to B/IB \leftarrow B) \rightsquigarrow B$ structure presheaf

$\overline{\mathcal{O}}_{\Delta} : (R \to B/IB \leftarrow B) \rightsquigarrow B/I$. "reduced structure presheaf"

cohomology of this is <u>Prismatic cohomology</u>, $\Delta_{R/A} \in D^{\geq 0}(A)$

cohomology of this is <u>Hodge-Tate cohomology</u>

$\overline{\Delta}_{R/A} \in D^{\geq 0}(\overline{A})$ (<u>no</u> canonical <u>maps</u>)

# Prismatic envelopes

**Lemma:** $(A, I)$ prism.

Forgetful functor $\underline{\mathrm{Prism}}_{(A,I)} \to \mathrm{Ring}_{\delta, A}$

has a left adjoint (prismatic envelope).

**Pf** $A \longrightarrow B \longleftarrow$ need to free:

(assume $I = (d)$)

$B$ = derived $(p, d)$-complete.

and $d$ is a non-zero-divisor

(i.e. $B[d] = 0$).

# Finite nonempty products in the prismatic site

$\Rightarrow$ existence of finite products in Prism$(A, I)$
(b/c Rings$^\delta$ has coproducts)

# The prismatic site has weakly final objects

- Also get weakly final objects.
  - Could be ? with free $\delta$-rings over $A$, etc.

Lift from $\bar{A}$ to $A$ is not canonical (everything else is functorial).

$= A/I$