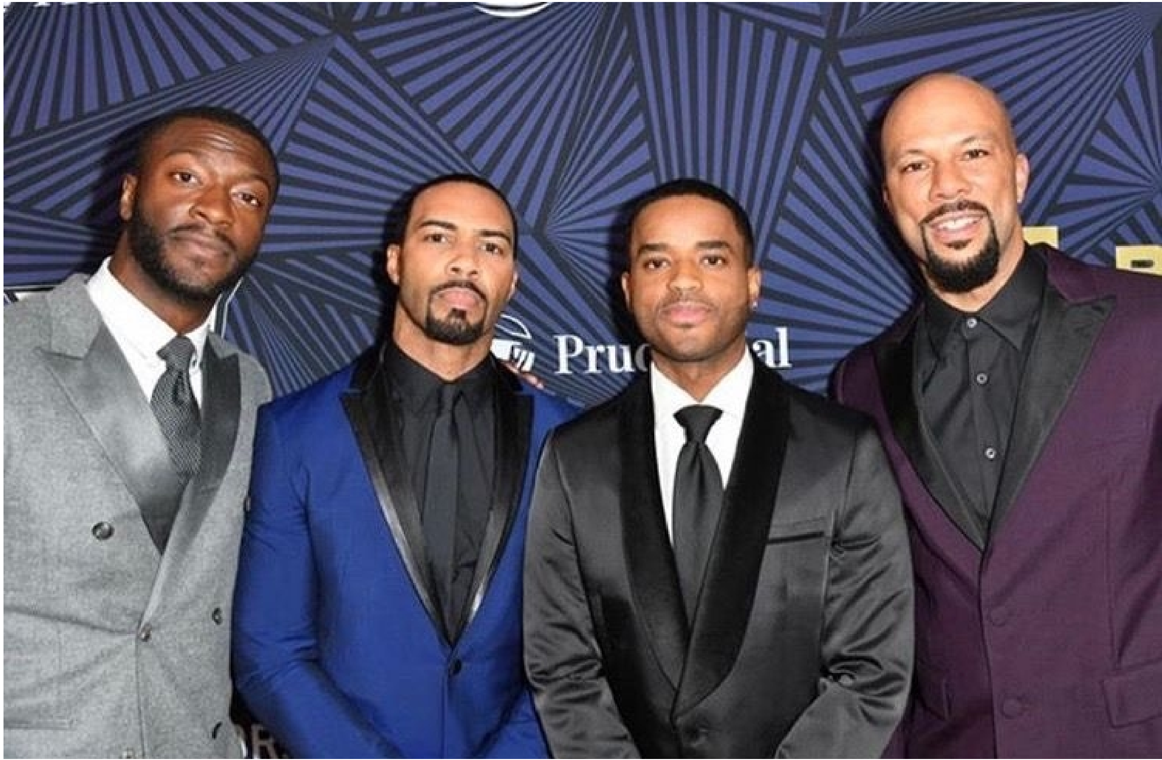
The Hodge-Tate comparison map

Aldus Hodge, Omari Hardwick, Larenz Tate, and Common (2017)

I decided to insert another background section, this one on double complexes (including totalization and the double complex spectral sequence). This will be covered this Friday (April 23).



Review: the relative prismatic site

(A, I) $\qquad \bar{A} = A/I$ (slice) (face)

$R = \bar{A}$-algebra

$$(R/A)_{\Delta} = \left\{ \begin{tikzcd} B/IB & B \arrow[l] \\ R \arrow[u] & \\ \bar{A} \arrow[u] & A \arrow[l] \arrow[uu, "\text{maps of prisms}"'] \end{tikzcd} \right\}$$

with indiscrete Grothendieck topology

(or flat topology)

$(A, I) \to (B, J)$ is faithfully flat if $A \to B$ is I-completely flat: $B \otimes^L_A A/I$ concentrated in deg 0 & faithfully flat over A/I.

has finite products, weakly final objects.

Remark: the case of a perfect prism

If (A, I) is a perfect prism ($\bar{A}$ = lens)

can consider subcategory of $(R/A)_{\Delta}$

where (B, IB) is also perfect

and B/I is p-normal

$\longrightarrow$ this is essentially the diamond of R

(Scholze)

B/I p-torsion free

& p-integrally closed in $B/I[1/p]$

($x \in B/I[1/p]$, $x^p \in B/I \Rightarrow x \in B/I$)

Graded commutativity

$E^{\bullet}$ = (not necessarily commutative) ≥ 0 graded rings

$E^{\bullet}$ is <u>graded commutative</u> if $a \in E^m$, $b \in E^n$

$$ab = (-1)^{mn} ba.$$

Example: $A \in$ Rings, $K^{\bullet}$ = commutative A-algebra object in $D(A)$

$$K^{\bullet} \otimes^{L}_{A} K^{\bullet} \xrightarrow{M} K^{\bullet}$$

Then $\bigoplus_{n \geq 0} H^n(K^{\bullet})$ inherits a <u>graded</u> commutative ring structure.

(has to do with $\mathrm{Tot}(L^{\bullet} \otimes M^{\bullet}) \cong \mathrm{Tot}(M^{\bullet} \otimes L^{\bullet})$)

Differential graded algebras (dgas)

$A \in \underline{\text{Ring}}$ $\quad$ A-dga is a complex $(E^\bullet, d)$ of A-modules in which $E^\bullet$ is also equipped with graded A-algebra structure subject to <u>signed Leibniz rule</u>

$$d^{n+m}(ab) = d^n(a)b + (-1)^n a d^m(b) \quad \begin{pmatrix} a \in E^n \\ b \in E^m \end{pmatrix}$$

<u>Commutative:</u> if $E^\bullet$ is graded commutative

<u>strictly commutative:</u> commutative + $a \cdot a = 0$ for any $a \in E^{\text{odd}}$.

Universal property of the (completed) de Rham complex

$A \to B$ morphism in $\underline{\text{Rings}}$

$(\Omega^{\bullet}_{B/A}, d_{dR}) = (B \to \Omega^1_{B/A} \to \Omega^2_{B/A} \to \cdots)$

<u>is strictly</u> commutative A-alg.

Let $(E^{\bullet}, d)$ be a graded <u>strictly commutative</u> A-alg
(in degrees ≥ 0)

Then $\eta : B \to E^0$ extends uniquely to $\Omega^{\bullet}_{B/A} \to E^{\bullet}$

in fact, can set by "ultimately commutative" <u>if</u>

$d(\eta(x))^2 = 0 \quad \forall x \in B$

$\in E^1$

Bockstein (Бокштейн) differentials: statement

A = rings, I = invertible, ideal

$M^\bullet \in D(A)$ $\quad \beta^n : H^n\Big(M \otimes^L_A \underbrace{I^n / I^{n+1}}_{(I/I^2)^{\otimes n}}\Big) \to H^{n+1}\big(M \otimes^L_A I^{n+1}/I^{n+2}\big)$

be the connecting homomorphism in

$$M^\bullet \otimes^L_A \left(0 \to I^{n+1}/I^{n+2} \to I^n/I^{n+2} \to I^n/I^{n+1} \to 0\right)$$

<u>Then</u> $\beta^{n+1} \circ \beta^n = 0$, so get a complex!

Bockstein (Бокштейн) differentials: proof

$$M \otimes^L_A$$

$$\begin{tikzcd}
0 \arrow[r] & I^{n+1}/I^{n+3} \arrow[r] \arrow[d] & I^n/I^{n+3} \arrow[r] \arrow[d] & I^n/I^{n+1} \arrow[r] \arrow[d] \arrow[ll, bend right] & 0 \\
0 \arrow[r] & I^{n+1}/I^{n+2} \arrow[r] & I^n/I^{n+2} \arrow[r] & I^n/I^{n+1} \arrow[r] \arrow[ll, bend left] & 0
\end{tikzcd}$$

$$0 \to I^{n+2}/I^{n+3} \to I^{n+1}/I^{n+3} \longrightarrow I^{n+1}/I^{n+2} \to 0$$

$$\beta^n$$

$$\beta^{n+1}$$

Bockstein differentials in Hodge-Tate cohomology

$\bar{\mathcal{O}}_{\Delta}$ sheaf on $(R/A)_{\Delta}$

use this instead of the ones in Bockstein construction

For $M \in \mathrm{Mod}_{\bar{A}}$

$M\{n\} = M \otimes_{A/I} I^n/I^{n+1}$

(Breuil-Kisin twists)

$\Rightarrow \quad \beta_I : H^n(\bar{\Delta}_{R/A})\{n\} \to H^{n+1}(\bar{\Delta}_{R/A})\{n+1\}$

$\bigoplus_i H^i(\bar{\Delta}_{R/A})\{i\}$ is a commutative $\bar{A}$-dga.

$\Rightarrow$ get $\Omega_{R/\bar{A}} \longrightarrow \bigoplus H^n(\bar{\Delta}_{R/A})\{n\}$

(modulo checking strict commutativity) map of $\bar{A}$-dgas.

Statement of the comparison theorem

Thm (A, I) bounded

R = p-completely smooth $\bar{A}$-algebra

(e.g. p-completion of smooth $\bar{A}$-algebras; in fact, by Elkik this is equivalent)

Then
$$\eta_R^\bullet : \Big(\underbrace{\widehat{\Omega}^\bullet_{R/\bar{A}}}_{\text{p-completed dR complex}}, d_{dR}\Big) \to \Big(H^\bullet(\bar{\Delta}_{R/A})\{\bullet\}, \beta_I\Big)$$

is an isomorphism of $\bar{A}$-dgas.

(note: $H^0(\bar{\Delta}_{R/A}) \leftarrow R$).

Example: q-de Rham cohomology of the torus

$$(A, I) = \left(\mathbb{Z}_p[[q-1]], \left([p]_q = \frac{q^p - 1}{q-1}\right)\right)$$

$$\bar{A} \cong \mathbb{Z}_p(\zeta_p) \quad q \mapsto \zeta_p$$

$$R = \bar{A}\langle X^{\pm}\rangle = p\text{-completion of } A[X^{\pm}]$$

$$= \widehat{\bigoplus_{i \in \mathbb{Z}}} \bar{A} X^i$$

We'll show (much later) $\Delta_{R/A} \cong \left(A\langle X^{\pm}\rangle \xrightarrow{\nabla_q} A\langle X^{\pm}\rangle \frac{dX}{X}\right)$

$$\cong \widehat{\bigoplus_{i \in \mathbb{Z}}} A X^i \xrightarrow{[i]_q} A X^i$$

Just one more thing...

$$(i)_q = \begin{cases} \text{unit in } \bar{A} \text{ if } i \not\equiv 0 \bmod p \\ \text{zero in } \bar{A} \text{ if } i \equiv 0 \bmod p. \end{cases}$$

$i = mr$

so $$\overline{\bar{\Delta}_{R/A}} \cong \widehat{\bigoplus_{k \in \mathbb{Z}}} \left(\bar{A}\, x^{kp} \xrightarrow{\;\sim\;} \bar{A}\, x^{kp}\right).$$

If $p = 2$, don't know a priori that HT complex is strictly commutative.

or even $d(\gamma(x))^2 = 0$ $x \in R$.

Proceed, compute HT complex of $R = \bar{A}\langle x \rangle$, see that you get 0 in degree 2, base change from this case.

Proof of the Bockstein lemma

Or can calculate directly!

use fact that $p=2$, so $\delta(a \smile b) = \delta(a) \smile \delta(b) + b(a \smile b)$

(alg assume $I = (f)$)

pick nearly final object $(F, I\text{-}\mathrm{tor}(R/A))\Delta$

where F is f-torsion-free.

Start computing $\eta(f)$ by lifting from $F_{I} f$ to $F \ldots$