Double complexes

Due to low usage, my Monday evening office hours (8-9pm PDT) will hereafter be by request only, via PM in Zulip. (If you're not on Zulip, use the Google Form on the course website to request access.)

Data Type Declarations

- Basic data types are:
  - `INTEGER` – integer numbers (+/-)
  - `REAL` – floating point numbers
  - `DOUBLE PRECISION` – extended precision floating point
  - `CHARACTER*n` – string with up to n characters
  - `LOGICAL` – takes on values `.TRUE.` or `.FALSE.`
  - `COMPLEX` – complex number
- Integer and Reals can specify number of bytes to use
  - Default is: `INTEGER*4` and `REAL*4`
  - `DOUBLE PRECISION` is same as `REAL*8`
- Arrays of any type must be declared:
  - `DIMENSION A(3,5)` – declares a 3 x 5 array (implicitly REAL)
  - `CHARACTER*30 NAME(50)` – directly declares a character array with 30 character strings in each element
- FORTRAN 90/95 allows user defined types

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A picture of a double complex

$\mathcal{A}$ = fixed abelian category

- diagram commutes
- every row and column is a complex

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & K^{p,q+1} \arrow[u] \arrow[r, "d_1^{p,q+1}"] & K^{p+1,q+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & K^{p,q} \arrow[u, "d_2^{p,q}"] \arrow[r, "d_1^{p,q}"] & K^{p+1,q} \arrow[u, "d_2^{p+1,q}"] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

bounded below

everything here is zero

Totalization

$K^{\bullet,\bullet}$ = double complex

$$\mathrm{Tot}^n(K^{\bullet,\bullet}) = \bigoplus_{p+q=n} K^{p,q}$$

$$d^n = \sum_{p+q=n} d_1^{p,q} + (-1)^p d_2^{p,q}$$

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & K^{p,q+1} \arrow[u] \arrow[r, "d_1^{p,q+1}"] & K^{p+1,q+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & K^{p,q} \arrow[u, "d_2^{p,q}"] \arrow[r, "d_1^{p,q}"] & K^{p+1,q} \arrow[u, "d_2^{p+1,q}"] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Interchanging the rows and columns

$K^{\bullet,\bullet}$ = double complex

transpose $L^{\bullet,\bullet}$ $\quad L^{p,q} = K^{q,p}$

There is natural isomorphism

$$\mathrm{Tot}(K^{\bullet,\bullet}) \cong \mathrm{Tot}(L^{\bullet,\bullet})$$

but identification of $K^{q,p} \subset \mathrm{Tot}^{p+q}(K^{\bullet,\bullet})$

with $L^{p,q} \subset \mathrm{Tot}^{p+q}(L^{\bullet,\bullet})$

is via multiplication by $(-1)^{pq}$

Graded commutativity

$A \in \underline{\text{Ring}}$ (bounded below)

$K^\bullet$ is commutative algebra object in $D(A)$.

interpret $K^\bullet \otimes^L K^\bullet \to K^\bullet$ as a map of complexes $\underbrace{L_1^\bullet \otimes L_2^\bullet}_{\text{viewed as a double complex}} \to L_3^\bullet$

ring structure on $H^\bullet(K^\bullet)$ comes from:

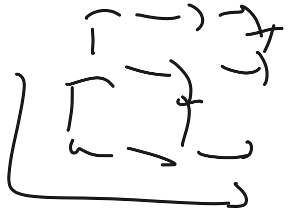
$$L_1^m \otimes L_2^n \to \mathrm{Tot}^{m+n}(L_1^\bullet \otimes L_2^\bullet) \to L_3^{m+n}$$

(changing order when identifying $\mathrm{Tot}(L_1^\bullet \otimes L_2^\bullet)$ with $\mathrm{Tot}(L_2^\bullet \otimes L_1^\bullet)$)

$\Rightarrow$ graded commutativity.

The spectral sequence(s) of a double complex

$K^{\bullet,\bullet}$ = double complex bounded below



$E_i^{p,q} \in \mathcal{A}$, $d_{(i)}^{p,q} : E_i^{p,q} \to E_i^{p+i,\, q+1-i}$

s.t. 1. $E_0^{p,q} = K_0^{p,q}$

2. $d_{(i)}^{p,q}$ form complexes (for each i)

3. $i=0$: $d_{(0)}^{p,q} = (-1)^p d_2^{p,q}$ of $K^{\bullet,\bullet}$

4. $E_{i+1}^{p,q}$ cohomology of $d_{(i)}^{p,q}$ i.e. $\dfrac{\ker d_{(i)}^{p,q}}{\operatorname{im}\left(d_{(i)}^{p-i,\, q-1+i}\right)}$

5. $i=1$: $d_{(1)}^{p,q}$ induced by $d^{p,q}$

... $\Rightarrow E_\infty^{p,q}$ for $p+q=n$ quotients of a filtration of $H^n(\mathrm{Tot}(K^{\bullet,\bullet}))$

The spectral sequence(s) of a double complex

Moreover, everything is natural:

$$K^{\bullet,\bullet} \longrightarrow L^{\bullet,\bullet}$$

then $E_i^{p,q}(K^{\bullet,\bullet}) \longrightarrow E_i^{p,q}(L^{\bullet,\bullet})$

compatible with differentials

Illustration

E_0 page

E_0 stage

E_0 sheet

$$\begin{tikzcd}
E_0^{0,3} & E_0^{1,3} & E_0^{2,3} & E_0^{3,3} \\
E_0^{0,2} \arrow[u, "d_{(0)}^{0,2}"] & E_0^{1,2} \arrow[u, "d_{(0)}^{1,2}"] & E_0^{2,2} \arrow[u, "d_{(0)}^{2,2}"] & E_0^{3,2} \arrow[u, "d_{(0)}^{3,2}"] \\
E_0^{0,1} \arrow[u, "d_{(0)}^{0,1}"] & E_0^{1,1} \arrow[u, "d_{(0)}^{1,1}"] & E_0^{2,1} \arrow[u, "d_{(0)}^{2,1}"] & E_0^{3,1} \arrow[u, "d_{(0)}^{3,1}"] \\
E_0^{0,0} \arrow[u, "d_{(0)}^{0,0}"] & E_0^{1,0} \arrow[u, "d_{(0)}^{1,0}"] & E_0^{2,0} \arrow[u, "d_{(0)}^{2,0}"] & E_0^{3,0} \arrow[u, "d_{(0)}^{3,0}"]
\end{tikzcd}$$

$$\begin{tikzcd}
E_1^{0,3} \arrow[r, "d_{(1)}^{0,3}"] & E_1^{1,3} \arrow[r, "d_{(1)}^{1,3}"] & E_1^{2,3} \arrow[r, "d_{(1)}^{2,3}"] & E_1^{3,3} \\
E_1^{0,2} \arrow[r, "d_{(1)}^{0,2}"] & E_1^{1,2} \arrow[r, "d_{(1)}^{1,2}"] & E_1^{2,2} \arrow[r, "d_{(1)}^{2,2}"] & E_1^{3,2} \\
E_1^{0,1} \arrow[r, "d_{(1)}^{0,1}"] & E_1^{1,1} \arrow[r, "d_{(1)}^{1,1}"] & E_1^{2,1} \arrow[r, "d_{(1)}^{2,1}"] & E_1^{3,1} \\
E_1^{0,0} \arrow[r, "d_{(1)}^{0,0}"] & E_1^{1,0} \arrow[r, "d_{(1)}^{1,0}"] & E_1^{2,0} \arrow[r, "d_{(1)}^{2,0}"] & E_1^{3,0}
\end{tikzcd}$$

$$\begin{tikzcd}
E_2^{0,3} \arrow[drr, "d_{(2)}^{0,3}"] & E_2^{1,3} \arrow[drr, "d_{(2)}^{1,3}"] & E_2^{2,3} & E_2^{3,3} \\
E_2^{0,2} \arrow[drr, "d_{(2)}^{0,2}"] & E_2^{1,2} \arrow[drr, "d_{(2)}^{1,2}"] & E_2^{2,2} & E_2^{3,2} \\
E_2^{0,1} \arrow[drr, "d_{(2)}^{0,1}"] & E_2^{1,1} \arrow[drr, "d_{(2)}^{1,1}"] & E_2^{2,1} & E_2^{3,1} \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0}
\end{tikzcd}$$

$$\begin{tikzcd}
E_3^{0,3} \arrow[ddrrr, "d_{(3)}^{0,3}"] & E_3^{1,3} & E_3^{2,3} & E_3^{3,3} \\
E_3^{0,2} \arrow[ddrrr, "d_{(3)}^{0,2}"] & E_3^{1,2} & E_3^{2,2} & E_3^{3,2} \\
E_3^{0,1} & E_3^{1,1} & E_3^{2,1} & E_3^{3,1} \\
E_3^{0,0} & E_3^{1,0} & E_3^{2,0} & E_3^{3,0}
\end{tikzcd}$$

Corollaries of the spectral sequence

<u>Corollary</u> If $\underline{K^{\bullet,\bullet} \longrightarrow L^{\bullet,\bullet}}$ is a morphism of double complexes and $E_i^{p,q}(K) \to E_i^{p,q}(L)$ is an isomorphism, then $\mathrm{Tot}(K^{\bullet,\bullet}) \to \mathrm{Tot}(L^{\bullet,\bullet})$
for any
some i
is a quasi-isomorphism.

<u>Pf</u>

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & 0 \\ & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \\ 0 & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & 0 \end{array}$$

five lemma $\Rightarrow$ middle vertical arrow is an isomorphism

Corollaries of the spectral sequence

Corollary Let $K^{\bullet,\bullet}$ be a double complex where (in degrees ≥ 0)

$K^{p,\bullet}$ is acyclic for all $p > 0$.

Then $K^{\bullet,0} \longrightarrow \mathrm{Tot}(K^{\bullet,\bullet})$

is a quasi-isomorphism

(get isom at E_1 page).

$$\begin{array}{ccccccc} K^0 & \to & K^1 & \to & K^2 & \to & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ M^{0,?} & \to & M^{1} & \to & M^{2} & \to & \end{array}$$

Collation of objects in D(A)

can use spectral sequence

justify claim:

given a sequence

$$0 \to K^{*,0} \to K^{*,1} \to \cdots \text{ in } D(A)$$

it has a well-behaved totalization in D(A)

What we are going to do with all of this

interested in add rows which are acyclic,

$$\begin{array}{ccccccc} F^{0,j} & \to & F^{1,j} & \to & F^{2,j} & \to & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ F^{0,j+1} & \to & F^{1,j+1} & \to & & & \end{array}$$

such that: all columns are quasi-isom to each other and rows induce maps of following form in $D(A)$

$$K^{\bullet} \xrightarrow{0} K^{\bullet} \xrightarrow{1} K^{\bullet} \xrightarrow{0} K^{\bullet} \xrightarrow{1} \cdots$$

(from simplicial structure)

$\Longleftrightarrow$ totalization is also quasi-isom to first column