Double complexes

Due to low usage, my Monday evening office hours (8-9pm PDT) will hereafter be by request only, via PM in Zulip. (If you're not on Zulip, use the Google Form on the course website to request access.)

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Data Type Declarations

- Basic data types are:
  - `INTEGER` – integer numbers (+/-)
  - `REAL` – floating point numbers
  - `DOUBLE PRECISION` – extended precision floating point
  - `CHARACTER*n` – string with up to n characters
  - `LOGICAL` – takes on values `.TRUE.` or `.FALSE.`
  - `COMPLEX` – complex number

- Integer and Reals can specify number of bytes to use
  - Default is: `INTEGER*4` and `REAL*4`
  - `DOUBLE PRECISION` is same as `REAL*8`

- Arrays of any type must be declared:
  - `DIMENSION A(3,5)` – declares a 3 x 5 array (implicitly REAL)
  - `CHARACTER*30 NAME(50)` – directly declares a character array with 30 character strings in each element

- FORTRAN 90/95 allows user defined types
A picture of a double complex

$A = \text{fixed abelian category}$

- diagram category
- every row and column is a complex

bounded below
Totalization

\[ \text{Tot}^\wedge (K^\otimes, \otimes) = \bigoplus K^{p, q} \]

\[ d^\wedge = \sum d_1^{p, q} + (1) d_2^{p, q} \]

\[ \cdots \rightarrow K^{p, q+1} \xrightarrow{d_1^{p, q+1}} K^{p+1, q+1} \rightarrow \cdots \]

\[ \cdots \rightarrow K^{p, q} \xrightarrow{d_2^{p, q}} K^{p+1, q} \rightarrow \cdots \]

\[ \cdots \rightarrow K^{p, q} \xrightarrow{d_1^{p, q}} K^{p+1, q} \rightarrow \cdots \]
Interchanging the rows and columns

\[ K^{0,-} = \text{double copy of } k \times \text{transpose } L^{0,-} \]

There is normalization \( T_{0,-}(K^{0,-}) \equiv T_{0,-}(L^{0,-}) \)

But identification of \( K^{0,-} \subset T_{0,-} \) with \( L^{0,-} \subset T_{0,-} \) is via multiplication by \((-1)^{\frac{d-1}{2}}\)
Graded commutativity

\[ A \in \text{Ring} \] (bounded below)

\[ K^0 \] is a \textit{Lie} algebra object in \( \mathcal{D}(A) \).

interpret \( K^0 \otimes K^0 \to K^0 \) as a map of complexes \( L_1 \otimes L_2 \to L_3 \)

via a double complex \( \text{Tot}(L_i \otimes L_j) \to L_3 \)

choosing an orientation (identity) \( \text{Tot}(L_i \otimes L_j) \) with

\( \text{Tot}(L_i \otimes L_j) \)

\( \Rightarrow \) graded commutativity.
The spectral sequence(s) of a double complex

\[ E^{p,q}_2 = \text{double complex} \quad \Rightarrow \quad \text{hocolim} \quad \Rightarrow \quad \text{holim} \]

\[ E^{p,q}_2 \subset A, \quad d^{p,q}_i : E^{p,q}_i \to E^{p+1, q+1-i}_i \]

1. \[ E^0 = E^{p,q}_0 \]

2. \[ d^{p,q}_i \text{ form complex} \quad \forall i \]

3. \[ d^{p,q}_i = (\text{anti}) \quad \text{for each} \quad i \]

4. \[ E^{p,q} \text{ columnwise of} \quad d^{p,q}_i \quad \Rightarrow \quad \text{hocolim} \]

5. \[ d^{p,q}_i \text{ are} \quad \text{induced by} \quad d^{p,q}_i \]

6. \[ E^{p,q}_\infty = \text{hocolim of} \quad E^{p,q}_2 \]

(1) \[ \Rightarrow \quad E^{p,q}_\infty \text{ for} \quad p+q \geq \text{filtration of} \]

\[ \Rightarrow \quad (\text{hocolim of} \quad d^{p,q}_i) \]

\[ \Rightarrow \quad (\text{holim of} \quad d^{p,q}_i) \]

\[ \Rightarrow \quad (\text{hocolim of} \quad d^{p,q}_i) \]
The spectral sequence(s) of a double complex

Moreover, every term is exact:

\[ K^{p,q} \rightarrow L^{p,q} \]

Then \( E_1^{p,q}(K^{\ast,\ast}) \rightarrow E_1^{p,q}(L^{\ast,\ast}) \)

compatible with differentials.
Corollaries of the spectral sequence

Corollary: If \( K^{*,*} \rightarrow L^{*,*} \) is a morphism of double complexes and \( E_{i,j}^r(K) \rightarrow E_{i,j}^r(L) \) is an isomorphism, then \( Tot(K^{*,*}) \rightarrow Tot(L^{*,*}) \) is a quasi-isomorphism.

**Proof:**

\[
\begin{array}{ccccccccc}
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]

The bottom row gives an exact sequence of kernels and cokernels.
Corollaries of the spectral sequence

Let $K^*$ be a complex where $K^p,^q$ is acyclic for all $p > 0$.

Then $K^*$ → $\text{Tot}(K^p,^q)$ is a quasi-isomorphism.

(set im at $E_1$ page).
Collation of objects in $D(A)$

(we use spectra if we need $N$
justify claim)

given a sequence

$0 \to K^n \to K^{n+1} \to \ldots \in D(U)$

it has a well-behaved triangulation

in $D(U)$
What we are going to do with all of this

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