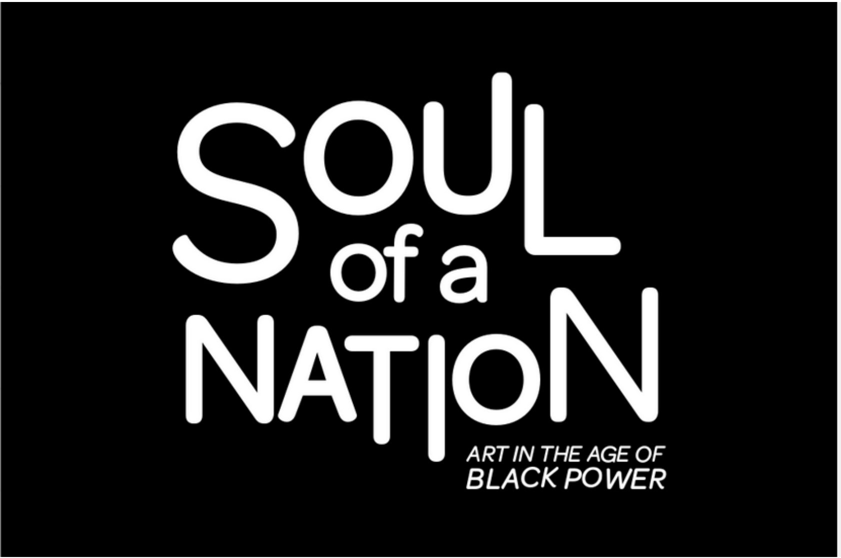
Hodge-Tate comparison for a crystalline prism

Monday evening 4 three hours by request only

$(A, I) = (A, pA)$ over $(\mathbb{Z}_p, p)$



Soul of a Nation: Art in the Age of Black Power

FEBRUARY 3 - APRIL 23, 2018

Soul of a Nation: Art in the Age of Black Power shines a bright light on the vital contribution of Black artists to an important period in American history and art. Featuring the work of 60 artists and including vibrant paintings, powerful sculptures, street photography, murals, and more, this landmark exhibition is a rare opportunity to see era-defining artworks that changed the face of art in America.

Developed by the Tate Modern in London and debuting in the US at Crystal Bridges, *Soul of a Nation: Art in the Age of Black Power* examines the influences, including the civil rights movement, Minimalism, and abstraction, on artists such as Romare Bearden, Noah Purifoy, Martin Puryear, Faith Ringgold, Betye Saar, Alma Thomas, Charles White, and William T. Williams.

Reminder: statement of the comparison theorem

(A, I) bounded prism (today $I = (p)$)

R = (p-completely) smooth $\bar{A}$-algebra $\quad \bar{A} = A/I$

(today $\bar{A}(x_1, \ldots, x_n)$)

Then $\eta^*_R : (\Omega^*_{R/\bar{A}}, d_{dR}) \longrightarrow (H^*(\bar{\Delta}_{R/A})\{*\}, \beta_I)$

is an isomorphism.

de Rham cohomology in characteristic p

$d(f^p) = pf^{p-1}df = 0.$

in char p!

i.e. For $R \to S$ in $\mathrm{Rings}_{\mathbb{F}_p}$, $\phi_S : \Omega^i_{S/R} \to \Omega^i_{S/R}$ is zero for all $i > 0$.

$R \in \mathrm{Rings}_{\mathbb{F}_p}$, $S = R[x_1, \dots, x_n]$

$\phi_{S/R} : S^{(1)} \to S$ relative Frobenius (i.e. $x_i \mapsto x_i^p$)

$S^{(1)} = S \otimes_{R, \phi_R} R.$

$$\begin{array}{ccc} S & \xrightarrow{\phi_S} & S \\ \uparrow & & \uparrow \\ R & \xrightarrow{\phi_R} & R \end{array}$$

Then $(\Omega_{S^{(1)}/R}, 0) \xrightarrow{\phi_{S/R}} (\Omega^\bullet_{S/R}, d_{dR})$ is a quasi-isom.

The Cartier isomorphism for affine space (and beyond)

Why is this? $\quad (\Omega^\bullet_{S^{(1)}/R}, 0) \longrightarrow (\Omega^\bullet_{S/R}, d_R)$

e.g. $S = R[x]$ $\quad (R[x] \xrightarrow{d} R[x]\,dx) \xrightarrow{x \mapsto x^p} (R[x] \xrightarrow{d} R[x]\,dx$

$$R[x] = \bigoplus_{i=0}^{p-1} x^i \underbrace{R[x^p]}_{\phi_{S/R}(S^{(1)})}$$

$\longrightarrow x^{i-1} R[x^p]\,dx$

for $i \neq 0$ action of $\frac{d}{dx}$ on $x^i R[x^p]$ is invertible

$x^i f(x^p) \longrightarrow i x^{i-1} f(x^p)\,dx$

$\longrightarrow$ something similar for $R \to S$ smooth.

Divided power operations (and an example)

$R \in \underline{\text{Ring}}$ that are R. The $\underline{\text{Divided power operations}}$

$\gamma_n : R \to R \otimes_{\mathbb{Z}} \mathbb{Q}$ are maps $\gamma_n(x) = \frac{x^n}{n!}$

Note: $\gamma_n(x+y) = \sum_{i=n} \gamma_i(x)\gamma_{n-i}(y)$

$\gamma_n(xy) = x^n \gamma_n(y)$.

$\gamma_m(\gamma_n(x)) = \frac{(mn)!}{(n!)^m m!} \gamma_{mn}(x)$ $\in R$

For $J \subset R$, say that $\underline{R \text{ has divided powers on } J}$ if $\gamma_n(x) \in R \; \forall x \in J$.

Divided power envelopes

e.g. $R = \mathbb{Z}_{(p)}$ admits divided powers on (p).

$\frac{p^n}{n!} \in \mathbb{Z}_{(p)}$ $(n \geq 1)$

R-Mat

R = rings, $J \subset R$ ideal

$D_J(R)$ = divided power envelope

= subring of $R \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by R and $\gamma_n(x)$ $x \in J$, $n \geq 0$

= minimal subring of $R \otimes_{\mathbb{Z}} \mathbb{Q}$ containing R admitting divided powers on J.

e.g. $(\mathbb{Z}[x], x) \to \left(\mathbb{Z}\left[\frac{x^n}{n!}\right]\right)$

Divided powers and the Poincaré lemma

$A \in \underline{\mathrm{Ring}}$ p-torsion-free $\quad P = A[x]$

$D = p$-adic completion of $D_{\underline{x}, p}(P)$. Then

$$d : D \longrightarrow D \otimes_P \widehat{\Omega}^1_{P/A} = D\, dx$$

is surjective with kernel A.

$$D = \widehat{A\left\langle \frac{x^n}{n!} \right\rangle}_{(p)}$$

$$d\left(\frac{x^{n+1}}{(n+1)!}\right) = \frac{x^n}{n!}\, dx.$$

(and likewise for multiple variables)

$\left(A \longrightarrow D \otimes_P \widehat{\Omega}^\bullet_{P/A} \text{ is quasi-isom.}\right)$

Divided powers on δ-rings: an exercise

Exercise: $R = \mathbb{Z}_{(p)}$-algebra admitting a δ-ring structure

Then R admits divided powers on J iff $\gamma_p(x) \in R \ \forall x \in J$.

Free δ-ring

$\Rightarrow R = \mathbb{Z}_{(p)}\{x\}$ and $J = xR$, the map $R \to D_J(R)$ promotes to $\underline{\mathrm{Ring}_\delta}$!

Pf ϕ extends from R to $D_J(R)$: $\phi(\gamma_n(x)) = \gamma_n(\phi(x))$

Check this gives a ful. lift:

$\phi(\gamma_n(x)) \equiv \gamma_n(x)^p \equiv 0 \pmod p \quad \forall n \geq 1$

A concrete divided power envelope

$$\Rightarrow \cancel{A}\, D_X\left(\mathbb{Z}_{(p)}\{X\}\right) = \mathbb{Z}_{(p)}\left\{X, \frac{\phi(X)}{p}\right\}$$

$$= \frac{\mathbb{Z}_{(p)}\{Z, X\}}{\delta\text{-ideal}(\phi(X) - pZ)} \qquad \mathbb{Z}_{(p)}\left\{X, \frac{\phi(X)}{p}\right\} \subseteq \mathbb{Q}\{X\}$$

(and similarly in multiple variables)

$$D_{X_1,\dots,X_n}\left(\mathbb{Z}_{(p)}\{X_1,\dots,X_n\}\right) = \mathbb{Z}_{(p)}\left\{X_i, \frac{\phi(X_i)}{p}\right\}$$

Hodge-Tate cohomology of affine space: setup

WTS: Hodge-Tate comparison for $(A, I) = (\mathbb{Z}_p, p)$

$R = \mathbb{F}_p[x_1, \dots, x_r]$.

idea: study $\overline{\Delta}_{R/A} \in D(\bar{A})$ using $\Delta_{R/A} \in D(A)$

$= \Delta_{R/A} \otimes^L_A \bar{A}$

(P, I_P) = weakly final obj of $(R/A)_\Delta$.

$P^n = \mathbb{Z}_p\{x_{ij} : i = 1, \dots, r;\ j = 0, \dots, n\}^\wedge$

$J : P^n \to R \to \bar{A}$

$$\begin{tikzcd}
0 \arrow[r] & P^0\{J^0/p\}^\wedge_{(p)} \arrow[r] \arrow[d] & P^1\{J^1/p\}^\wedge_{(p)} \arrow[r] \arrow[d] & P^2\{J^2/p\}^\wedge_{(p)} \arrow[r] \arrow[d] & \cdots \subset \Delta_{R/A} \\
0 \arrow[r] & P^0\{\phi(J^0)/p\}^\wedge_{(p)} \arrow[r] \arrow[d, equal] & P^1\{\phi(J^1)/p\}^\wedge_{(p)} \arrow[r] \arrow[d, equal] & P^2\{\phi(J^2)/p\}^\wedge_{(p)} \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & D_{J^0}(P^0) \arrow[r] & D_{J^1}(P^1) \arrow[r] & D_{J^2}(P^2) \arrow[r] & \cdots
\end{tikzcd}$$

$\downarrow \phi_A = \text{isom.}$

Hodge-Tate cohomology of affine space: rows

$$
\begin{tikzcd}
D_{J^0}(P^0) \arrow[r] \arrow[d] & D_{J^1}(P^1) \arrow[r] \arrow[d] & D_{J^2}(P^2) \arrow[r] \arrow[d] & \cdots \\
D_{J^0}(P^0)\widehat{\otimes}_{P^0}\widehat{\Omega}^1_{P^0/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^1}(P^1)\widehat{\otimes}_{P^1}\widehat{\Omega}^1_{P^1/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^2}(P^2)\widehat{\otimes}_{P^2}\widehat{\Omega}^1_{P^2/\mathbb{Z}_p} \arrow[r] \arrow[d] & \cdots \\
D_{J^0}(P^0)\widehat{\otimes}_{P^0}\widehat{\Omega}^2_{P^0/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^1}(P^1)\widehat{\otimes}_{P^1}\widehat{\Omega}^2_{P^1/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^2}(P^2)\widehat{\otimes}_{P^2}\widehat{\Omega}^2_{P^2/\mathbb{Z}_p} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}
$$

acyclic!

acyclic!

Hodge-Tate cohomology of affine space: columns

all resolve the same thing by Poincaré, so

1 0 1 1 1 0

$$
\begin{tikzcd}
D_{J^0}(P^0) \arrow[r] \arrow[d] & D_{J^1}(P^1) \arrow[r] \arrow[d] & D_{J^2}(P^2) \arrow[r] \arrow[d] & \cdots \\
D_{J^0}(P^0)\widehat{\otimes}_{P^0}\widehat{\Omega}^1_{P^0/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^1}(P^1)\widehat{\otimes}_{P^1}\widehat{\Omega}^1_{P^1/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^2}(P^2)\widehat{\otimes}_{P^2}\widehat{\Omega}^1_{P^2/\mathbb{Z}_p} \arrow[r] \arrow[d] & \cdots \\
D_{J^0}(P^0)\widehat{\otimes}_{P^0}\widehat{\Omega}^2_{P^0/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^1}(P^1)\widehat{\otimes}_{P^1}\widehat{\Omega}^2_{P^1/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^2}(P^2)\widehat{\otimes}_{P^2}\widehat{\Omega}^2_{P^2/\mathbb{Z}_p} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}
$$

we use later to relate to HT.