Hodge-Tate comparison for crystalline prisms (continued)

I’ve updated the web site with projected topics for the next few lectures. But this is subject to frequent revision, e.g., I already pushed the calendar back a day so that I could spend today revisiting the Hodge-Tate comparison for crystalline prisms.
Relative Frobenius \( R \to \mathfrak{s} \) morphism in \( \mathfrak{rs}_{1} \). 

Lemma: For \( i \geq 1 \), \( \phi_{S/R} : \mathfrak{S}_{S/R} \to \mathfrak{S}_{S/R}^{1} \) is zero 

(only using this when \( S = \mathbb{A}^{X, -\infty X} \) relative Frobenius.

\[
\begin{array}{c}
\text{Spec } S \\
\downarrow \phi_{S/R} \quad \downarrow \quad \phi_{S}^{*} \\
\text{Spec } S^{(1)} & \quad \rightarrow & \text{Spec } S \\
\downarrow & & \downarrow \\
\text{Spec } R & \phi_{R}^{*} & \rightarrow \text{Spec } R 
\end{array}
\]
Cartier isomorphism for affine space: the map

\[
\mathbb{A}^n(\mathbb{R}) \cong \mathbb{R}^n
\]

Then for \( \psi \) as \( \psi \) as above, the map

\[
\psi(\xi_1, \ldots, \xi_n) \mapsto (\sum_{i=1}^n \xi_i \cdot x_i)
\]

of \( \psi \)-dga as a \( \phi \)-algebra in degree 0.

\( \mathcal{L} \) free on \( dx_{\xi_1}, \ldots, dx_{\xi_n} \), \( 1 \leq j, < \cdots < j_i \leq n \),

\( dx_{j_i} \mapsto x_j \cdot dx_{j_i} \).
Cartier isomorphism for affine space: proof

The case \( r = 1 \):

\[
\begin{align*}
(R(x_i) \to R(x_i) dx_1) & \cong (R((x_i) \to R(x_i) dx) \\
R(x_i) & \to x_1 \mapsto x_i^n \\
x_i & \to x_i^p, \ dx_1 \to x_i^p dx_1
\end{align*}
\]

\[
X^e : R(x_i^p) \to x_1^e R(x_i^p) dx \\
R = 0
\]

\( f \to x_0 \), this differential is invertible!

\[
X \left[ f(x_i^p) dx_1 \right] \mapsto e X, x_1^{e-1} f(x_i^p) dx.
\]
Claim: a map is canonical up to homotopy

\[ R(C^\vee_1 \otimes \cdots \otimes C^\vee_n) = R(C^\vee_1 \otimes \cdots \otimes C^\vee_n) \]

\[ \lambda(x_1 + x_2) \to (x_1 + x_2)^{p-1} \quad \text{if } (x_1 + x_2) \neq 0 \]

\[ = \frac{1}{p-1} \int_0^{x_1 + x_2} (x_1 + x_2)^{p-2} \, dx \]

(This will follow a posteriori from use of Cartier's hom in proof of Hodge-Tate comparison extension to \( S = \text{smooth R. algebra} \).)
A weakly final object (corrected) \( (A, I) = (\mathbb{Z}_p, (p)) \)

Need a weakly initial object \( R = \mathbb{F}_p \langle x_1, \ldots, x_n \rangle \) of \((\mathbb{Z}/m)^{op}\).

Define \( P = \mathbb{Z}_p \langle x_1, \ldots, x_n \rangle \overset{\phi}{\rightarrow} R \\
\phi(x_i) \rightarrow 0 \).

\[ J = \text{ker} \ (P \rightarrow R) \]

\[ (B, \phi_\beta) \subset (\mathbb{Z}/m)^{op} \overset{A}{\rightarrow} R \overset{B}{\rightarrow} (\beta) \]

Choose \( B \)

\[ \mathbb{Z}_p \langle x_1, \ldots, x_n \rangle \overset{\mathbb{Z}_p}{\rightarrow} B \]

\[ \mathbb{Z}_p \langle x_1, \ldots, x_n \rangle \overset{\phi}{\rightarrow} B \text{ and } \]

The \( (P, \left \{ J \right \}^n, \phi) \)

is a weakly final object.
A Čech-Alexander complex

\[ p^{n} = R \{ x_{i,j} \}_{i,j}^{(p)} \]

\[ J^{n} = \ker ( p^{n} \to R ) \quad x_{i,j} \mapsto x_{i} \]

\[ \Delta R/A \simeq \]

\[ (0 \to p^{0} \langle J^{0} / p \rangle^{(p)} \to p^{1} \langle J^{1} / p \rangle^{(p)} \to \ldots) \]
Enter divided power envelopes

\[ \phi : A \to A \text{ is a resolution.} \]

\[ A \to P^\bullet \]

\[ 0 \to P^0 \{ J^0/p \}_{(p)} \to P^1 \{ J^1/p \}_{(p)} \to P^2 \{ J^2/p \}_{(p)} \to \cdots \]

\[ \text{Compute} \quad \Delta_{P/A} \]

\[ 0 \to P^0 \{ \phi(J^0)/p \}_{(p)} \to P^1 \{ \phi(J^1)/p \}_{(p)} \to P^2 \{ \phi(J^2)/p \}_{(p)} \to \cdots \]

\[ \text{Completed divided power map} \quad \ell(x) = (\mathbb{Z}_p \times x, x^d \otimes \phi(x)) \]

\[ 0 \to D_{j^0}(P^0)_{(\phi)} \to D_{j^1}(P^1)_{(\phi)} \to D_{j^2}(P^2)_{(\phi)} \to \cdots \]
Claim: The totalization of this double complex is quasi-isomorphic to the column with the $0$th column.
The rows: a combinatorial lemma about differentials

For \( Q = \text{a polynomial ring over} \mathbb{R} \)

For in the complex

\[
\mathbb{C} \to \mathbb{C} \to \cdots
\]

is any \( \mathcal{A} \) (in fact homotopic to \( \mathbb{R} \) at level of \( Q \)-cosimplicial modules).

reduces to \( i = 1 \), \( Q = \mathbb{R}[x] \)

some combinatorial statement.
The columns: multiple resolutions of a single object

\[ K \xrightarrow{\frac{1}{\Delta}} K \xrightarrow{\beta} K \xrightarrow{\gamma} K \xrightarrow{\delta} K \xrightarrow{\epsilon} \cdots \]

\[ \text{\textcircled{K}} \xrightarrow{\theta} (K \xrightarrow{\alpha} K) \circ (K \xrightarrow{\beta} K) \circ \cdots \]
Extracting the Hodge-Tate isomorphism

So far, $\Delta R A = \text{crystalline Cohomology of a three space.}$

$\Delta R A \cong \Delta R / A \otimes_{\mathbb{Z}_p} \mathbb{F}_p$

Have an isomorphism

$\left(\mathbb{L}^0 R^1\left(1 A, \right)\right) \cong \left( H^* \left( \Delta R A, \mathbb{F}_p \right) \right)$

Note: This map can be computed in terms of Bockstein differential (in this particular case).