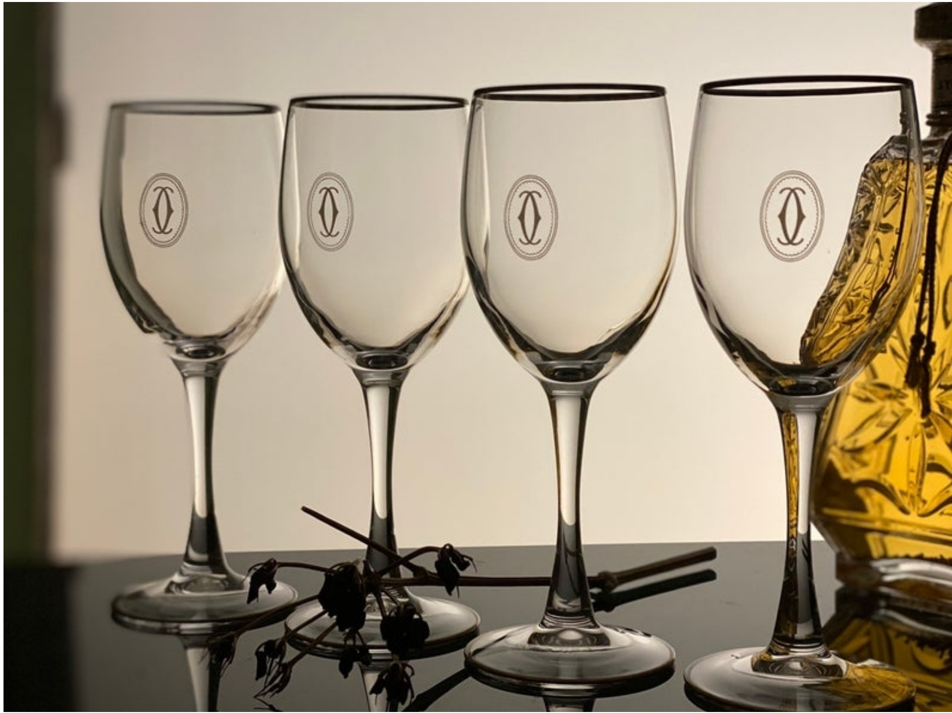
Hodge-Tate comparison for crystalline prisms (continued)

I've updated the web site with projected topics for the next few lectures. But this is subject to frequent revision, e.g., I already pushed the calendar back a day so that I could spend today revisiting the Hodge-Tate comparison for crystalline prisms.



Relative Frobenius

$R \to S$ morphism in Rings$/\mathbb{F}_p$

Lemma: For $i \geq 1$, $\phi_{S/R} : \Omega^i_{S^{(1)}/R} \to \Omega^i_{S/R}$ is zero

(only using this when $S = R[x_1, \ldots, x_r]$)

relative Frobenius.

$$
\begin{tikzcd}
\operatorname{Spec} S \arrow[dr, dashed, "\phi^*_{S/R}"] \arrow[drr, bend left, "\phi^*_S"] \arrow[ddr, bend right] & & \\
& \operatorname{Spec} S^{(1)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec} S \arrow[d] \\
& \operatorname{Spec} R \arrow[r, "\phi^*_R"] & \operatorname{Spec} R
\end{tikzcd}
$$

Cartier isomorphism for affine space: the map

$R \in \mathrm{Rings}_{\mathbb{F}_p}$ $\quad S = R[x_1, \dots, x_r]$

$\phi_{S/R}: S^{(1)} \to S$ $\qquad$ Then $\exists$ quasi-isom

$\cong$ $\quad x_i \mapsto x_i^p$

$R(x_1, \dots, x_r)$

$$(\Omega^\bullet_{S^{(1)}/R}, 0) \to (\Omega^\bullet_{S/R}, d_{dR})$$

of $S^{(1)}$-dgas acting as $\phi_{S/R}$ in degree 0.

Ω^i free on $dx_{j_1} \wedge \dots \wedge dx_{j_i}$ $\qquad 1 \le j_1 < \dots < j_i \le r$

$$dx_j \longmapsto x_j^{p-1}\, dx_j$$

Cartier isomorphism for affine space: proof

The case $r=1$:

$$\left(R[x_1] \xrightarrow{d} R[x_1]\,dx_1\right) \simeq \left(R[x_1] \to R[x_1]\,dx\right)$$

$x_1 \mapsto x_1^p$

$x_1 \mapsto x_1^p,\ dx_1 \mapsto x_1^{p-1}\,dx_1$

$$\bigoplus_{e=0}^{p-1} x_1^e R[x_1^p] \to x_1^{e-1} R[x_1^p]\,dx$$

For $e \neq 0$, this differential is invertible!

$$x_1^e f(x_1^p) \xmapsto{d} e\,x_1^{e-1} f(x_1^p)\,dx.$$

Canonicality of the Cartier isomorphism

Claim: Cartier map is canonical up to homotopy

i.e.

$$R(x_1, \ldots, x_n) = R(y_1, \ldots, y_n)$$

$$d(x_1 + x_2) \longrightarrow (x_1 + x_2)^{p-1} d(x_1 + x_2)$$

$$= x_1^{p-1} dx + x_2^{p-1} dx$$

$$+ (\text{something homotopic to zero})$$

(this will follow a posteriori from use of Cartier isom in proof of Hodge-Tate comparison)

$\rightsquigarrow$ extends to S = smooth R-algebra.

A weakly final object (corrected) $(A, I) = (\mathbb{Z}_p, (p))$

Need a weakly initial object $R = \mathbb{F}_p(x_1, \dots, x_r)$ of $(R/A)^{op}_{\Delta}$:

$$P = \mathbb{Z}_p\{x_1, \dots, x_r\}^{\wedge}_{(p)} \longrightarrow R$$

$$\delta^n(x_i) \longrightarrow 0.$$

$J = \ker(P \to R)$

Then $\left(P\{\tfrac{J}{p}\}^{\wedge}_{(p)}, (p)\right)$ is a weakly final object.

$(B, pB) \in (R/A)^{op}_{\Delta}$ $\quad R \to B/pB$

can choose

$\mathbb{Z}_p[x_1, \dots, x_r] \to B$ of rings

$\mathbb{Z}_p\{x_1, \dots, x_r\} \to B$ of δ-rings

P

A Čech-Alexander complex

$$P^n = \mathbb{Z}_p\{x_{ij}\}^{\wedge}_{(p)} = (\overbrace{P \otimes_{\mathbb{Z}_p} \cdots \otimes_{\mathbb{Z}_p} P}^{n+1})^{\wedge}_{(p)}$$

$i = 1 \ldots r$

$j = 0 \ldots n$

$$J^n = \ker(P^n \to R) \qquad x_{ij} \to x_i$$

$$\delta(x_{ij}) \to 0$$

Thm $\Delta_{R/A} \simeq$

$$\left(0 \to P^0\{J^0/p\}^{\wedge}_{(p)} \to P^1\{J^1/p\}^{\wedge}_{(p)} \to \cdots\right)$$

$$\overline{\Delta}_{R/A} = \Delta_{R/A} \otimes^{L}_{\mathbb{Z}_p} \mathbb{F}_p$$

Enter divided power envelopes

and crystalline cohomology

$\phi : A \to A$ isomorphism.

$A \to P^\bullet$ is a cosimplicial resolution

computes $\Delta_{R/A}$

compute $\Delta_{R/A}$

p-completed divided power envelope of $(\mathbb{Z}_p[x], x)$ is $\mathbb{Z}_p\{x, \frac{\phi(x)}{p}\}$

$$
\begin{tikzcd}
0 \arrow[r] & P^0\{J^0/p\}^\wedge_{(p)} \arrow[r] \arrow[d] & P^1\{J^1/p\}^\wedge_{(p)} \arrow[r] \arrow[d] & P^2\{J^2/p\}^\wedge_{(p)} \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & P^0\{\phi(J^0)/p\}^\wedge_{(p)} \arrow[r] \arrow[d, equal] & P^1\{\phi(J^1)/p\}^\wedge_{(p)} \arrow[r] \arrow[d, equal] & P^2\{\phi(J^2)/p\}^\wedge_{(p)} \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & D_{J^0}(P^0)^\wedge_{(p)} \arrow[r] & D_{J^1}(P^1)^\wedge_{(p)} \arrow[r] & D_{J^2}(P^2)^\wedge_{(p)} \arrow[r] & \cdots
\end{tikzcd}
$$

Total(ization) recall

Claim: the totalization of this double complex is q-isom to 0th row and 0th column.

from previous diagram,

totalizes to 0

$$\begin{tikzcd}
D_{J^0}(P^0) \arrow[r] \arrow[d] & D_{J^1}(P^1) \arrow[r] \arrow[d] & D_{J^2}(P^2) \arrow[r] \arrow[d] & \cdots \\
D_{J^0}(P^0)\widehat{\otimes}_{P^0}\widehat{\Omega}^1_{P^0/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^1}(P^1)\widehat{\otimes}_{P^1}\widehat{\Omega}^1_{P^1/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^2}(P^2)\widehat{\otimes}_{P^2}\widehat{\Omega}^1_{P^2/\mathbb{Z}_p} \arrow[r] \arrow[d] & \cdots \\
D_{J^0}(P^0)\widehat{\otimes}_{P^0}\widehat{\Omega}^2_{P^0/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^1}(P^1)\widehat{\otimes}_{P^1}\widehat{\Omega}^2_{P^1/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^2}(P^2)\widehat{\otimes}_{P^2}\widehat{\Omega}^2_{P^2/\mathbb{Z}_p} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

The rows: a combinatorial lemma about differentials

For Q = a polynomial ring over $\mathbb{Z}_p$

For $i>0$, the complex

$$\Omega^i_Q \to \Omega^i_{Q\otimes Q} \to \Omega^i_{Q\otimes Q\otimes Q} \to \cdots$$

is acyclic (in fact homotopic to zero as levelwise Q°-cosimplicial modules).

reduce to $i=1$, $Q=\mathbb{Z}_p[x]$

some combinatorial statement.

The columns: multiple resolutions of a single object

Proposition: each column is resolution of $\mathbb{Z}_p(0)$. In particular, all quasi-isom.

&

$$K^\bullet \underset{1}{\overset{1}{\rightrightarrows}} K^\bullet \overset{1}{\underset{1}{\overset{1}{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}}}} K^\bullet \longrightarrow \cdots$$

$$K' \overset{\sim ?}{\xrightarrow{\;0\;}} \left(K^0 \xrightarrow{\;1\;} K^1\right) \xrightarrow{\;0\;} \left(K^0 \xrightarrow{\;1\;} K^1\right) \longrightarrow \left(\cdots\right.$$

Extracting the Hodge-Tate isomorphism

So far, $\Delta_{R/A}$ = crystalline cohomology of affine space.
(i.e. $\mathbb{Z}_p\langle x\rangle$)

$$\overline{\Delta}_{R/A} \simeq \Delta_{R/A} \otimes^{L}_{\mathbb{Z}_p} \mathbb{F}_p$$

have an isomorphism

$$(\Omega^{\bullet}_{R^{(1)}/A}, \partial) \simeq (H^{\bullet}(\overline{\Delta}_{R/A}), \beta_{I})$$

WTS: Cartier map can be computed in terms of Bockstein differential (in this particular case)