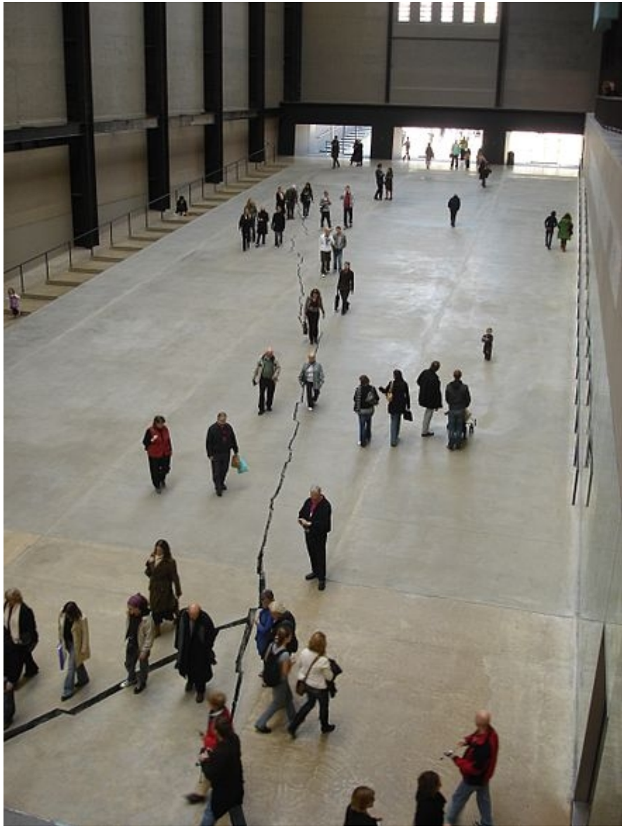
Proof of the Hodge-Tate comparison

Shibboleth, Tate Modern, 2007



Statement of the comparison theorem

(A, I) = bounded prism $\qquad \bar{A} = A/I$ slice

R = p-completely smooth $\bar{A}$-algebra

Bockstein-Kisin twists

Then $\eta_R^{\bullet}: (\hat{\Omega}^{\bullet}_{R/\bar{A}}, d_R) \to (H^{\bullet}(\bar{\Delta}_{R/A})\{\bullet\}, \beta_I)$

Bockstein differential

is an isomorphism in $D(R)$.

An explicit computation in the crystalline case

Last time: $(A, I) = (\mathbb{Z}_p, (p))$ $R = \mathbb{F}_p[x_1, \ldots, x_n]$

Goal: $\Delta_{R/A} \cong$ crystalline cohomology of affine space

$\to$ reduced mod p to dR com.

But is this the right map? check in degree 1.

class of $x_i \in H^0(\Delta_{R/A})$ maps via β_0 to $\dfrac{x_{i,0} - x_{i,1}}{p}$

$$\boxed{0 \to \frac{p\Gamma}{p^2\Gamma} \to \frac{\Gamma}{p^2\Gamma} \to \frac{\Gamma}{p\Gamma} \to 0}$$

$\Delta_{R/A}$ x_i

$\phi^* \Delta_{R/A}$

$\dfrac{x_{i,0} - x_{i,1}}{p}$

$$\begin{tikzcd}
0 \arrow[r] & P^0\{J^0/p\}^\wedge_{(p)} \arrow[r] \arrow[d, "\phi"] & P^1\{J^1/p\}^\wedge_{(p)} \arrow[r] \arrow[d, "\phi"] & P^2\{J^2/p\}^\wedge_{(p)} \arrow[r] \arrow[d, "\phi"] & \cdots \\
0 \arrow[r] & P^0\{\phi(J^0)/p\}^\wedge_{(p)} \arrow[r] \arrow[d, equal] & P^1\{\phi(J^1)/p\}^\wedge_{(p)} \arrow[r] \arrow[d, equal] & P^2\{\phi(J^2)/p\}^\wedge_{(p)} \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & D_{J^0}(P^0) \arrow[r] & D_{J^1}(P^1) \arrow[r] & D_{J^2}(P^2) \arrow[r] & \cdots
\end{tikzcd}$$

$\dfrac{x_{i,0} - x_{i,1}}{p}$

$x_i^{p-1} dx_i$

$x_{i,0}^{p-1} dx_{i,0} - x_{i,1}^{p-1} dx_{i,1}$

$$\begin{tikzcd}
D_{J^0}(P^0) \arrow[r] \arrow[d] & D_{J^1}(P^1) \arrow[r] \arrow[d] & D_{J^2}(P^2) \arrow[r] \arrow[d] & \cdots \\
D_{J^0}(P^0)\widehat{\otimes}_{P^0}\widehat{\Omega}^1_{P^0/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^1}(P^1)\widehat{\otimes}_{P^1}\widehat{\Omega}^1_{P^1/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^2}(P^2)\widehat{\otimes}_{P^2}\widehat{\Omega}^1_{P^2/\mathbb{Z}_p} \arrow[r] \arrow[d] & \cdots \\
D_{J^0}(P^0)\widehat{\otimes}_{P^0}\widehat{\Omega}^2_{P^0/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^1}(P^1)\widehat{\otimes}_{P^1}\widehat{\Omega}^2_{P^1/\mathbb{Z}_p} \arrow[r] \arrow[d] & D_{J^2}(P^2)\widehat{\otimes}_{P^2}\widehat{\Omega}^2_{P^2/\mathbb{Z}_p} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

Étale localization for HT cohomology

Reminder: $R \to S$ is smooth iff locally on $\mathrm{Spec}(S)$ it factors as $R \to R[x_1, \dots x_r] \xrightarrow{\text{étale}} S$.

Lemma: Let $R \to S$ be p-completely étale map between p-completely smooth $\bar{A}$-algebras. Then $\overline{\Delta}_{R/A} \hat{\otimes}^L_R S \longrightarrow \overline{\Delta}_{S/A}$ is an isomorphism.

Key point: can fill in the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ B \arrow[r] & * \\ B \arrow[u] \arrow[r] & O\,?? \arrow[u] \end{tikzcd}$$

need p-completely étale morphism here. $\to B$

Base change for prismatic and HT cohomology

$(A,I)\to(A',I')$ morphism of prisms

s.t. $A\to A'$ has finite p-complete Tor amplitude

R = p-completely smooth $\bar{A}$-algebra.

For $R'=R\hat{\otimes}_A A'$, we have natural isoms.

$$\Delta_{R/A}\hat{\otimes}^L_A A'\cong\Delta_{R'/A'},\quad \bar{\Delta}_{R/A}\hat{\otimes}^L_A A'\cong\bar{\Delta}_{R'/A'}.$$

bounded below complex. for some universal constant c

i.e. for any $M\in D^{\geq 0}_{comp}(A)$ $M\hat{\otimes}^L_A A'\in D^{\geq -c}_{comp}(A)$

A remark about non-crystalline prisms

Say $I = (d)$ $\quad P = A[x] \in \mathrm{Ring}_\delta$ $\quad \delta(x) = 0$

The derived (p, d)-completion of $P\{\frac{\phi(x)}{d}\}$ is not

the derived (p,d)-completion of pd-envelope of

$(P, (x))$. $\quad$ E.g. $(A, (d)) = (\mathbb{Z}_p[[q-1]], [p]_q)$

in this case, replace pd-envelope with

"q-pd-envelope".

The universal oriented prism

$A_0 = \mathbb{Z}_{(p)}\{d\}$ $S =$ mult subset generated by $\delta(d)$, $\varphi^n(\delta(d))$

$A_1 = S^{-1} A_0$ now (A, I) $I = (d)$

$A = \hat{A_1}_{(p,d)}$ is a prism (oriented)

bounded (p-torsion-free)

(p, d) is a regular sequence.

$\Rightarrow B = A\left\{\frac{\varphi(d)}{p}\right\}^{\wedge}_{p}$ is classically (p, d) completion of

pd-envelope at $(A, (d))$.

(also p-torsion-free)

Base change for the universal prism

$\alpha : A \to B \xrightarrow{\phi_B} B$

in B, $\phi(d)$ is distinguished and $p \mid \phi(d)$ $\Rightarrow$ p is distinguished.

so we get a map of prisms

$$(A,(d)) \to (B,(d)) \xrightarrow{\phi_B} (B,(\phi(d))) = (B,(p))$$

crystalline!

$A/p \to B/p$ factors as $A/p \to A/(p,d) \xrightarrow{\phi} A/(p,d^p) \to B/p$

finite p-complete Tor-amplitude $[-1]$

faithfully flat (Kunz)

Reduction from the universal prism to a crystalline prism

$\alpha : A \xrightarrow{\text{ét}} A \to B.$ $\quad \alpha^* : D_{comp}(A) \to D_{comp}(B)$

$\searrow B \to \mathbb{Z}_p$ is conservative (reflects isomorphisms)

and $\alpha^* \Delta_{R/A} \cong \Delta_{R_B/B}$ $\quad R_B = R \hat{\otimes}_A B$

(b/c finite Tor-amplitude)

$\Rightarrow$ transfer HT isomorphism from

(B/p) crystalline case (get this from $(\mathbb{Z}_p, (p))$ by base change.

to (A, d) case!

Reduction from a general prism to the universal prism

Now: if (A, I) is general (oriented) and map from universal to (A, I) has finite Tor-amplitude

$\Rightarrow$ use base change to recover HT isom.

to deal with general case, do

universal

$$\begin{tikzcd}
 & A_0 \arrow[r] \arrow[d] & A \arrow[d] \\
R_p \arrow[r] & D_0 \arrow[r] & A \otimes^L_{A_0} D_0
\end{tikzcd}$$

simplicial rings.

Another transfer via Witt vectors

$\Delta_{R/A} = ?$

$A \to \bar{A}$ in Rings ← mixed char Witt vrs!!

$A \to W(\bar{A})$ in Rings$_\delta$.

$\psi : A \to W(\bar{A}) \xrightarrow{\phi} W(\bar{A})$

carries I into (p):

$I \longrightarrow V(W(\bar{A})) \longrightarrow \phi \circ V(W(\bar{A})) = pW(\bar{A})$

so I get a map $(A, I) \to (W(\bar{A}), (p))$ of prisms

provided that the target is a prism

(i.e. p-torsion-free)

Crystalline and de Rham comparison

Thm (A, I) bounded and $W(\bar{A})$ p-torsion-free

R = p-completely smooth $\bar{A}$-algebra

Then $\exists$ natural isom $\Delta_{R/A} \hat{\otimes}^{L}_{A,\phi} \bar{A} \cong \Omega^{\wedge}_{R/\bar{A}}$.

idea get this from a comparison of $\Delta_{R/A}$ with crystalline cohomology valued in $W(\bar{A})$.