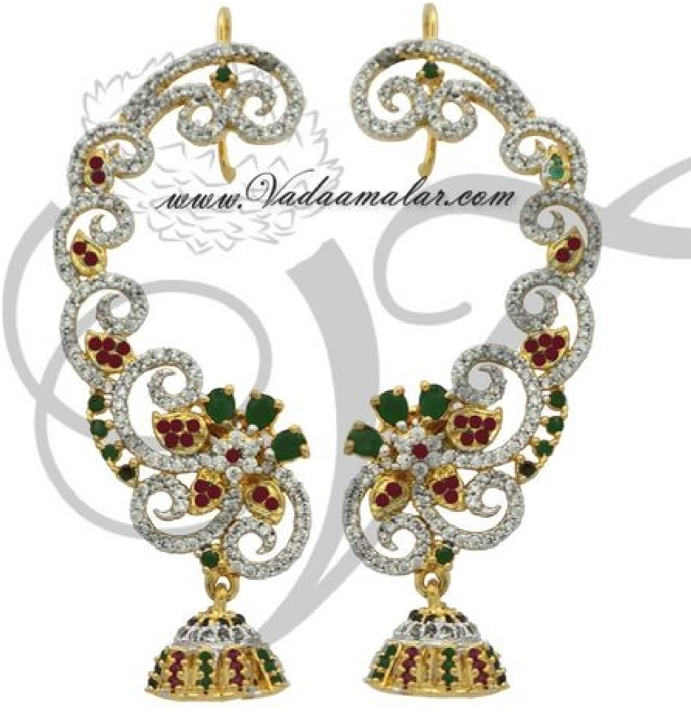
# Nonabelian derived functors



# Review: simplicial and cosimplicial objects in a category

$\Delta$ = category of finite nonempty ordered sets

$$[0] \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; [1] \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; [2] \;\cdots$$

A cosimplicial object in a category $\mathcal{C}$ is a covariant functor $U : \Delta^{\cancel{op}} \longrightarrow \mathcal{C}$

simplicial:
$$X_2 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X_1 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X_0$$

# Simplicial and cosimplicial resolutions

A simplicial resolution of $X \in \mathcal{C}$ is a simplicial object $U : \Delta^{op} \to \mathcal{C}$ with colimit $X$.

$$\left( \cdots U_2 \; \substack{\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow} \; U_1 \; \substack{\rightarrow\\[-2pt]\leftarrow\\[-2pt]\rightarrow} \; U_0 \right) \to X$$

e.g. if $\mathcal{C} = \mathrm{Mod}_A$, then get a resolution

$$\cdots \to U_2 \to U_1 \to U_0 \to 0$$

map $U_{n+1} \to U_n$ is alternating sum of $n+2$ arrows in the picture.

# A trivial (?) example

trivial resolution

$$X \rightleftarrows X \rightleftarrows X$$

$X \in \mathcal{C}$

then $\mathcal{C} = \mathrm{Mod}_A$ set associated complex

$$\cdots \to M \xrightarrow{1} M \xrightarrow{0} M \xrightarrow{1} M \xrightarrow{0} M \to 0$$

$$\downarrow \wr$$

$$M[\partial].$$

# Homotopies and homotopy equivalences

$\mathcal{C}$ assumed to have finite (co)products.

$U, V$ = simplicial objects of $\mathcal{C}$ $\quad a, b : U \to V$

A homotopy from $a$ to $b$ is a morphism

$$h : U \times \Delta[1] \to V \qquad \begin{aligned} a &= h \circ e_0 \\ b &= h \circ e_1 \end{aligned}$$

$e_0, e_1$
$U$

$$\begin{tikzcd} U \arrow[r, "a"] \arrow[d] & V \arrow[d, phantom, sloped, "\cong"] \\ U \arrow[r, "b"] & V \end{tikzcd}$$

generate an equivalence relation.

$a, b$ homotopy inverses: $a \circ b$, $b \circ a$ equivalent to respective identities.

# Standard resolutions: a functorial description

$V : \mathcal{C}_1 \to \mathcal{C}_2$ has left adjoint $U : \mathcal{C}_2 \to \mathcal{C}_1$

$\eta : \mathrm{id}_2 \to V \circ U$, unit, $\quad \varepsilon : U \circ V \to \mathrm{id}_{\mathcal{C}_1}$ counit.

$X_n = \overbrace{U \circ V}^{n+1 \text{ times}} : \mathcal{C}_1 \to \mathcal{C}_1 \qquad X_{-1} = \mathrm{id}_{\mathcal{C}_1}$

Claim: apply $X_n$ to an object $Y \in \mathcal{C}$ to get a simplicial resolution where face maps by applying

$$U(\delta_j^n) = \mathrm{id}_{X_{j-1}} \star \varepsilon \star \mathrm{id}_{X_{n-j-1}}$$

$(n) \to (n+1)$ $\qquad U(\sigma_j^n) = \mathrm{id}_{\cancel{X}_{j-1} \circ V} \star \eta \star \mathrm{id}_{U \circ X_{n-j-1}}$

# Standard resolutions for modules

$\mathcal{C}_1 = \underline{\mathrm{Mod}}_A \qquad \mathcal{C}_2 = \underline{\mathrm{Set}} \qquad M \in \mathcal{C}_1$

$$\cdots \quad \begin{matrix} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{matrix} \quad \bigoplus_{m \in \bigoplus_{m \in M} A} A \quad \begin{matrix} \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{matrix} \quad \bigoplus_{m \in M} A \xrightarrow{\pi} M$$

$$\begin{matrix} \psi & & \\ [x] & \longrightarrow & [\pi(x)] \\ \| & & \text{or} \\ \left[\sum a_i [m]\right] & \longrightarrow & \sum a_i [m] \end{matrix}$$

# Standard resolutions for rings

$$\underline{\mathrm{Rings}}_A \to \underline{\mathrm{Set}}$$

$$B \in \underline{\mathrm{Rings}}_A$$

$$\pi : A(B) \to B$$

$$\begin{array}{ccc} A(A(B)) & \rightrightarrows & A(B) \\ & \leftarrow & \\ [m] & \longrightarrow & [\pi(m)] \\ [m] & \longrightarrow & m \\ \\ [[b]] & \longleftarrow & [b] \end{array}$$

# Left Kan extensions

...along this 'embedding'

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "F"] \arrow[dr, "G"'] & \mathcal{C}_2 \arrow[d, dashed, "L"] \arrow[dl, phantom, "\Uparrow\alpha"] \\ & \mathcal{C}_3 \end{tikzcd}$$

functor to be extended...

Given $M : \mathcal{C}_2 \to \mathcal{C}_3$

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & L \circ F \arrow[d, dashed, "\sigma \circ F"] \\ & M \circ F \end{tikzcd}$$

$\exists(!)\sigma :$

$L \Rightarrow M$

# Derived functors as a left Kan extension

$G : \underline{Mod}_A \to \mathcal{A}$ right exact

additive

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[dr, "G"'] & K^-(A) \arrow[d, dashed] \\
 & D(\mathcal{A})
\end{tikzcd}$$

$\mathcal{C}$ = complexes of projective modules

left Kan extension is left derived functor

$LG : K^-(A) \to D(\mathcal{A})$

# Setup for derived functors on rings

$A \in \underline{\text{Rings}}$ $\qquad$ $\underline{\text{Poly}}_A$ = category of polynomial algebras over $A$ in finitely many vars.

$F : \underline{\text{Poly}}_A \to D(\underline{Ab})$

Thm $F$ always admits a left Kan extension

$LF : \underline{\text{Rings}}_A \to D(\underline{Ab})$

- commutes with filtered colimits
- can be computed using simplicial resolutions,