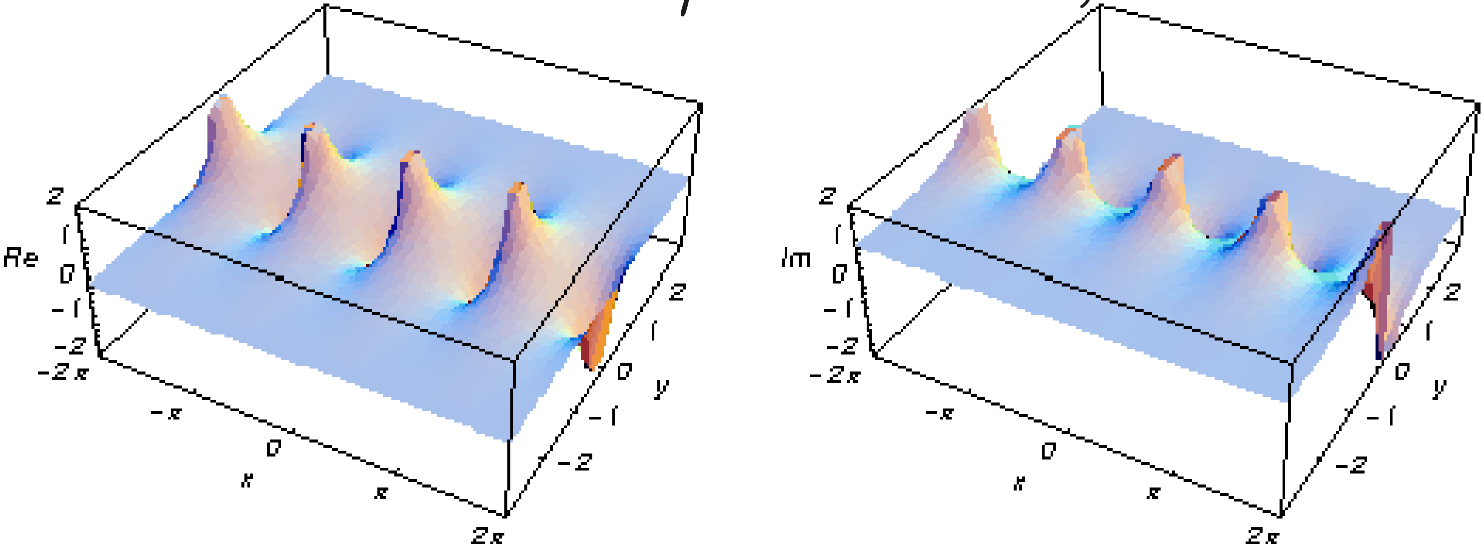
Applications of nonabelian derived functors

I've updated the section on nonabelian derived functors; I will briefly go over the main edits today, but read the notes for more details.

complex cotangent



From last time: products of simplicial sets

V = simplicial set (each $V_n = V([n])$ is finite nonempty)

$\mathcal{U}$ = simplicial object in $\mathcal{C}$ (admits finite coproducts)

$\mathcal{U} \times V$ = simplicial obj in $\mathcal{C}$

$$(\mathcal{U} \times V)_n = \coprod_{v \in V_n} \mathcal{U}_n$$

$\phi : [m] \to [n]$

$$\coprod_{v \in V_n} \mathcal{U}_n \to \coprod_{v \in V_m} \mathcal{U}_m$$

acts as $\mathcal{U}(\phi)$ on each factor.

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example: "n-simplex"

$$\Delta[n] : [k] \to \mathrm{Hom}_\Delta([k],[n])$$

$\mathcal{U} \times \Delta[1]$ "$= \mathcal{U}[1]$"

$e_0, e_1 : \mathcal{U} \to \mathcal{U} \times \Delta[1]$

From last time: homotopies of simplicial maps

Assume $\mathcal{C}$ has finite coproducts

homotopy from $a: U \to V$

to $b: U \to V$

is a morphism $h: U \times \Delta[1] \to V$

s.t. $h \circ e_0 = a$

$h \circ e_1 = b$.

generate equivalence relation

e.g. $\Delta(0) \to \Delta(n)$ is a homotopy equivalence.

any map.

From last time: left Kan extensions

$$
\begin{tikzcd}
\mathcal{C}_1 \arrow[r, "F"] \arrow[d, "G"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & \mathcal{C}_2 \arrow[d, dashed, "L"] \\
\mathcal{C}_3 \arrow[r, equal] & \mathcal{C}_3
\end{tikzcd}
$$

$$
\begin{tikzcd}
G \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & L \circ F \arrow[d, dashed, "\sigma \circ F"] \\
 & M \circ F
\end{tikzcd}
$$

$\forall$ $\exists!$

"is the most"

From last time: left derived functors on rings

Prop $A \in \underline{\text{Ring}}$ $F : \underline{\text{Poly}}_A \to D(\underline{Ab})$ covariant

Then F admits a left Kan extension

also works for simplicial rings.

$LF : \text{Ring}_A \to D(\underline{Ab})$

which is an extension (α is a natural isom.)

· LF commutes with filtered colimits

· Given $P_\bullet \to B$ simplicial resolution,

$$LF(B) = \operatorname{colim} LF(P^\bullet)$$

easy examples: $\Lambda^i : \text{Mod}_A \to \text{Mod}_A \xrightarrow{(c)} D(A)$.

The cotangent complex

$A \in \underline{\mathrm{Ring}}$

$L_{\cdot/A} : \underline{\mathrm{Ring}}_A \to D(A)$ extending $\underline{\mathrm{Poly}}_A \to D(A)$

in fact $L_{B/A} \in D(B)$ $\qquad B \to \Omega^1_{B/A}(0)$

- and even $L_{B/A} \in D^{\leq 0}(B)$

- $H^0(L_{B/A}) = \Omega^1_{B/A}$

- if $A \to B$ smooth, then $L_{B/A} = \Omega^1_{B/A}(0)$
  (just like for a polynomial ring)

Properties of the cotangent complex

In particular, if $A \to B$ étale then $L_{B/A} = 0$, but not conversely: e.g.

if $A \to B$ is a morphism of ~~perfect~~ $\mathbb{F}_p$-algebras, then again $L_{B/A} = 0$. (check for $B = A[x_i^{p^{-\infty}}]$)

— $A \to B \to C$ any morphisms of rings.

$$L_{B/A} \otimes^L_B C \to L_{C/A} \to L_{C/B} \to$$

is a distinguished triangle.

More properties of the cotangent complex

- $A \to B$ surjective, kernel I
  then $H^{-1}(L_{B/A}) = I/I^2$.

  Moreover, if I = generated by a regular sequence,
  then $H^i(L_{B/A}) = 0$ for $i \neq 0, -1$.

- $A \to B$, $A \to C$: $L_{C/A} \otimes^{L}_{A} B = L_{C \otimes^{L}_{A} B / B}$

simplicial tensor product!

Derived de Rham cohomology (in characteristic *p* only!)

$K \in \mathrm{Rings}_{\mathbb{F}_p}$ derived de Rham functor

$dR_{-/K} : \mathrm{Rings}_K \to D(K)$ is left derived of $\mathrm{Poly}_K \to D(K)$

$R \to \Omega^\bullet_{R/K}$.

warning: be careful about filtrations

(esp. if you want this in char 0).

Will define a certain increasing, exhaustive filtration on $dR_{R/K}$

(i.e. $\mathrm{Fil}_0 \to \mathrm{Fil}_1 \to \cdots$ in $D(K)$ with colimit $dR_{R/K}$.

associated graded are $C(\mathrm{Fil}_{i-1} \to \mathrm{Fil}_i)$

mapping cone.

Filtrations in derived categories

$\mathrm{Fil}_0 \to \mathrm{Fil}_1 \to \cdots \to$ with colimit $dR_{R/k}$

$gr_i(\mathrm{Fil}_\bullet) = \mathrm{Cone}(\mathrm{Fil}_{i-1} \to \mathrm{Fil}_i)$

The derived Cartier isomorphism

conjugate filtration

For $k \in \mathrm{Rings}_{\mathbb{F}_p}$, $R \in \mathrm{Rings}_{/k}$

$\exists$ functorial (in R) increasing exhaustive filtration on $dR_{R/k}$ in $D(R^{(1)})$

with

$$\mathrm{gr}_i\, dR_{R/k} \simeq \wedge^i L_{R^{(1)}/K}(-i).$$

Pf true in polynomial case (by usual Cartier;

reminder: Cartier for polynomial rings is (coordinate-independently by comparison to prismatic)

de Rham and derived de Rham in the smooth case

Cor For R smooth over k,

$$dR_{R/K} \cong \Omega^\bullet_{R/K}$$ (just like for poly rings).

pf use derived Cartier to reduce to corresponding fact about cotangent complex

$\Rightarrow dR_{\bullet/K}$ compatible with étale localization

Regular semiperfect rings

$k \in \mathrm{Rings}_{\mathbb{F}_p}$ perfect

Regular semiperfect k-algebra

is a ring of form $S = R/I$ for some $R \in \mathrm{Rings}_{\mathbb{F}_p}$ perfect

I = ideal gen by a regular sequence.

e.g. $k[x_1^{p^{-\infty}}, \ldots, x_r^{p^{-\infty}}] / (x_1, \ldots, x_r)$

$= k[x_1^{p^{-\infty}}, \ldots, x_n^{p^{-\infty}}] / (x_1, \ldots, x_r)$

derived de Rham cohomology of regular semiperfect rings

Lemma $k \in \underline{\text{Rings}}_{\mathbb{F}_p}$ perfect

$S \in \underline{\text{Rings}}_k$ regular semiperfect

$\Rightarrow$ $d\Omega_{S/k}$ is concentrated in degree 0!!

(i.e. $\in \underline{\text{Rings}}_k$!!!)

What ring is it???

pf easy.