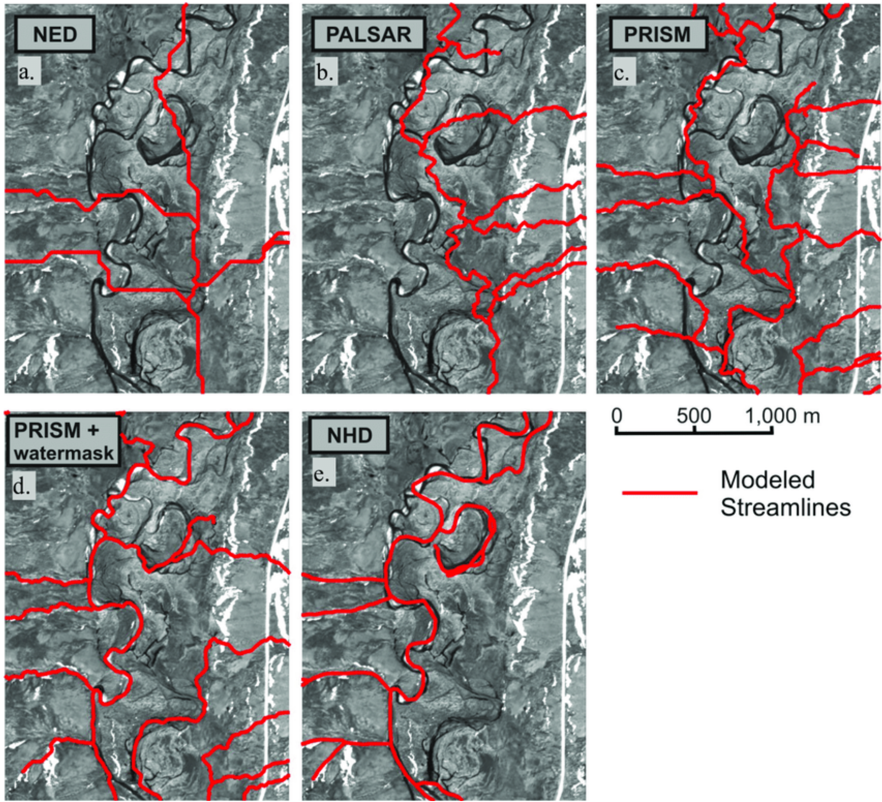
Derived prismatic cohomology



Derived crystalline cohomology

$k \in \mathrm{Ring}_{\mathbb{F}_p}$ perfect, $A \in \underline{\mathrm{Ring}}$ classically p-complete

$A/p \cong k$

$R\Gamma_{crys} : \underline{\mathrm{Poly}_k} \to D(A)$ $\quad R = k[x_1, \dots x_r]$

(using work on Hodge-Tate comparison, identify this with $\phi_A^* \Delta_{\tilde{R}/A}$)

$\underline{I}$

$\widetilde{\Omega}_{\tilde{R}/A}$ $\quad \tilde{R} = A[x_1 \dots x_r]$

$\rightsquigarrow$ left Kan extension, $\quad R\Gamma_{dcrys} : \underline{\mathrm{Ring}_{/k}} \to D(A)$

$$R\Gamma_{dcrys}(\cdot/A) \otimes^L_A k \simeq dR_{\cdot/k}.$$

Derived prismatic and Hodge-Tate cohomology

(A, I) bounded prism $\quad \bar{A} = A/I$ $\quad$ (p, I)-complete

derived prismatic cohomology $\quad L\Delta_{-/A} : \mathrm{Ring}_{\bar{A}} \to D_{comp}(A)$

Left Kan extension of $\mathrm{Poly}_{\bar{A}} \to D_{comp}(A)$

$R_0 \longrightarrow \widehat{\Delta_{R_0/A}}$ $\quad$ p-complete

has action of ϕ_A
is a "derived δ-ring"

p-completion.

derived Hodge-Tate cohomology $\quad L\bar{\Delta}_{-/A} : \mathrm{Ring}_{\bar{A}} \to D_{comp}(\bar{A})$

Left Kan extension of $\mathrm{Poly}_{\bar{A}} \to D_{comp}(\bar{A})$

$R_0 \longrightarrow \bar{\Delta}_{\widehat{R_0}/A}$.

both have ring structure

and $\quad L\Delta_{R/A} \otimes^L_A \bar{A} \cong L\bar{\Delta}_{\widehat{R_0}/A}$

The derived Hodge-Tate comparison theorem

(A, I) bounded prism, $R \in \underline{\mathrm{Ring}}_{\bar{A}}$

$L\bar{\Delta}_{R/A}$ admits a functorial filtration

$\mathrm{Fil}^{HT}_{\bullet}$ in $D_{comp}(\bar{A})$ s.t. $gr_i^{HT}(L\bar{\Delta}_{R/A}) =$

$$\wedge^i_{\bar{A}} L_{R/\bar{A}}\{i\}[-i]^{\wedge}_{(p)}$$

Proof: true for polynomial rings as before.

deduce the rest formally.

(and also: A = p-complete with bounded p-torsion, $A \to B$ flat $\Rightarrow$ B = classical p-completion, $L_{B/B}$ becomes 0 on p-completion).

Comparison with the smooth case

(A, I) bounded R = p-completely smooth $\bar{A}$-algebra.

Then $L\Delta_{R/A} \to \Delta_{R/A}$ is an isom.

Pf Same argument as for derived de Rham.

Warning: Hereafter I will usually write $\Delta_{R/A}$ for derived constructions.

$\bar{\Delta}_{R/A}$

Semilenses

Last time: defined a regular semiperfect k-algebras ($k \in \mathrm{Ring}_{\mathbb{F}_p}$ perfect) and stated $dR_{S/k}$ is discrete (in degree 0)

perfect

Lens = slice of a perfect prism (A, I)

$\bar{A}$

semilens = derived p-complete quotient of some lens.

S semilens $\Rightarrow S/p$ semiperfect and $\theta: W(S^\flat) \twoheadrightarrow S$.

$\overline{[L]}\Delta_{S/A} \in D^{\leq 0}(A)$ (see next slide)

$S = R/(f_1 \ldots f_r)^{-\bar{1}}$

$L_{R/k} = 0 \qquad k \to R \to S$

$L_{S/k} \xrightarrow{\sim} L_{S/R} \in D(S)$

$\overset{"}{I}/I^2[1]$

$(\wedge^i L_{S/R})[-i] = \wedge^i L_{S/R}{}^{(-i)}$ in degree 0

Prismatic cohomology of regular semilenses R

Regular semilens = quasi-lens S that is (locally) (regular, $\bar{I}$ generated)

(assume $\bar{A}$ p-torsion-free, S = p-completely flat / $\bar{A}$)

$\Rightarrow$ (L) $\bar{\Delta}_{S/A}$ admits a filtration with gradeds

$\wedge^i L_{S/A}\{-i\}[-i]$ ← finite projective S-modules.

concentrated in degree 0.

is p-completely flat S-algebra.

$\Rightarrow$ $\Delta_{S/A}$ concentrated in degree 0, $(p, \bar{I})$-completely flat over A

$(\Delta_{S/A}, I)$ is a better prism!

$\Delta_{S/A} \cong W(R^\flat)\{J/d\}^\wedge_{(p)}$.

Coperfection in characteristic p

$R \in \underline{\text{Ring}}_{\mathbb{F}_p}$ coperfection of R is colim R

$$= \text{colim } R \xrightarrow{\phi} R \xrightarrow{\phi} R \to \cdots$$

$k \in \underline{\text{Rings}}_{\mathbb{F}_p}$ perfect, $R \in \underline{\text{Rings}}_k$

$$(R^{(1)})_{perf} \xrightarrow{\sim} R_{perf}$$

<u>Comment</u> <u>Kunz criterion</u> R noetherian,
R regular iff $\phi_R : R \to R$ is flat.

For R arbitrary, $\phi_R : R \to R$ is flat
iff $R \to R_{perf}$ flat.

Coperfection via derived de Rham cohomology

k = perfect field of char p

$R \in \underline{\text{Rings}_k}$. Then $dR_{R/k} \to R$

induces an isomorphism of

$$dR_{R/k,perf} = \mathrm{colim}\left(dR_{R/k} \xrightarrow{\phi} dR_{R/k} \to \cdots \right)$$

with R_{perf}

---

pf True for poly rings: $\mathrm{colim}_{i>0}\, \Omega^i_{R/k} \xrightarrow{\phi} \Omega^i_{R/k} \to \cdots = 0$

(each map is 0)

Coperfection via derived Hodge-Tate cohomology

$(A, I) = (W(k), (p))$

Thm $\overline{\Delta}_{R/A, perf} = \mathrm{colim}(\overline{\Delta}_{R/A} \xrightarrow{\varphi} \overline{\Delta}_{R/A} \to \cdots)$

is discrete and isom to R_{perf}.

Pf Reduce to case $R = k[x_1 \dots x_r]$

check that "functorial"

$gr_i^{HT}(\varphi_R) : gr_i^{HT}(\overline{\Delta}_{R/A}) \to gr_i^{HT}(\overline{\Delta}_{R/A})$

coincides with "prismatic" Frobenius.

(check $R = k[x]$ by hand)

Coperfection via derived prismatic cohomology

$$\Delta_{R/A,\mathrm{perf}} = \boxed{\mathrm{colim}\left(\Delta_{R/A} \xrightarrow{\varphi_R} \Delta_{R/A} \xrightarrow{\varphi_R} \cdots\right)}$$

is discrete, isom to $W(R_{\mathrm{perf}})$

Next time: repeat in mixed characteristic
(i.e. for a non-crystalline prism)
and get ... something interesting.