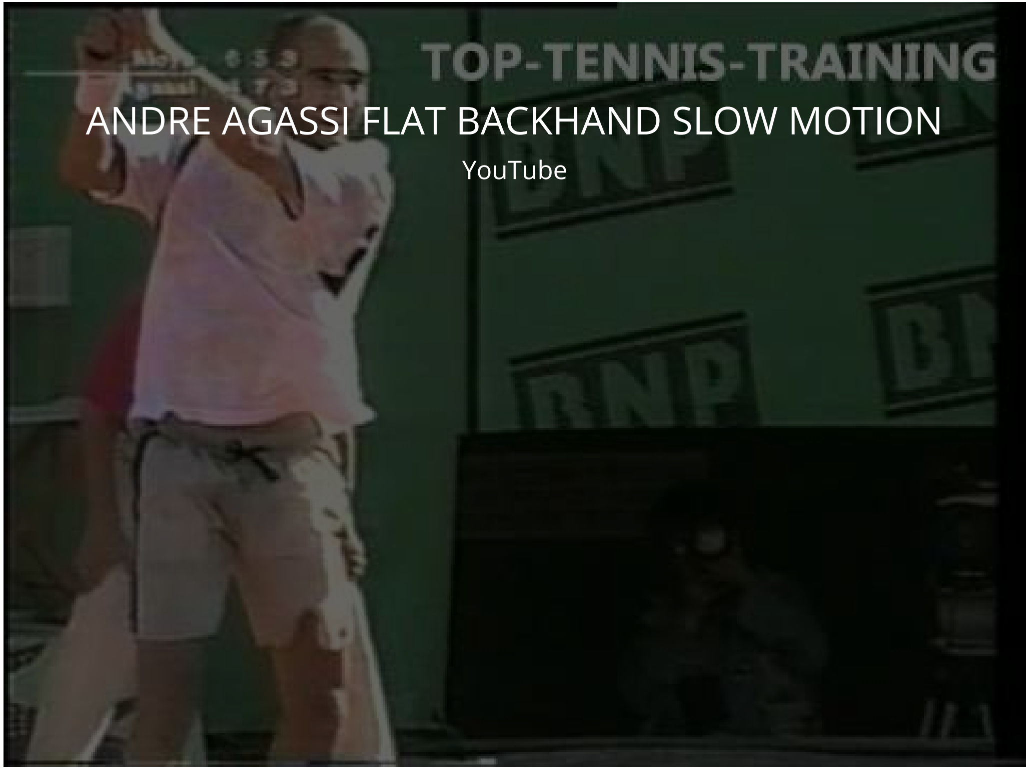
# Coperfections in mixed characteristic and applications



# Prismatic and lens coperfection

$(A, I)$ = perfect prism

$R$ = derived $p$-complete $\bar{A}$-algebra

<u>prismatic coperfection</u>

$$\Delta_{R/A,\mathrm{perf}} = \left(\operatorname{colim}\left(\Delta_{R/A} \xrightarrow{\phi} \cdots\right)\right)^{\wedge}_{(p,I)} \in D_{\mathrm{comp}}(A)$$

<u>lens coperfection</u>

$$R_{\mathrm{lens}} := \Delta_{R,A\mathrm{perf}} \otimes^{L}_{A} \bar{A} \in D_{\mathrm{comp}}(R)$$

$p$-complete

($R$-lin via $R \to \bar{\Delta}_{R/A} \xrightarrow{\sim} \Delta_{R/A,\mathrm{perf}} \otimes^{L}_{A} \bar{A}$

$\bar{\Delta}_{R/A} \otimes^{L}_{A} \bar{A} \nearrow$

$\phi_R : \Delta_{R/A,\mathrm{perf}}$ ⟲

both have ring object structure.

# Base independence of lens coperfection

Let $(A, I) \to (B, J)$ be a morphism of perfectoid prisms, $S \in \mathrm{Ring}_{\bar{B}}$ derived $p$-complete.

1. $\overline{\Delta}_{S/A} \to \overline{\Delta}_{S/B}$ is an isom.
2. If $S = \bar{B}$ then $S \xrightarrow{\sim} \overline{\Delta}_{S/B}$
3. $\Delta_{S/A} \xrightarrow{\sim} \Delta_{S/B}$ $\Rightarrow$ $\Delta_{S/A,\mathrm{perf}} \xrightarrow{\sim} \Delta_{S/B,\mathrm{perf}}$
   $\Rightarrow$ same lens coperfections

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$1,2 \Leftarrow$ Hodge-Tate comparison, use $\widehat{L_{\bar{B}/\bar{A}}}_{(p)} = 0$.

$1 \Rightarrow 3$ Derived Nakayama

## Coprefection for a crystalline prism

$$(A, \bar{I}) = \text{crystalline} = (W(\bar{A}), (p))$$

$$\left.\begin{aligned} \Delta_{R/A, \mathrm{perf}} &= W(R_{\mathrm{perf}}) \\ R_{\mathrm{lens}} &= R_{\mathrm{perf}} \end{aligned}\right\} \text{ in degree } 0.$$

# Coperfection for a lens

$R$ = lens $\quad R \in \mathrm{Rings}_{\bar{A}} \quad (A, I)$

$\Delta_{R/A} \cong W(R^b)$ concentrated in degree 0 $\quad$ perfect prism

$\cong$

$\Delta_{R/A, \mathrm{perf}}$ $\quad$ placed in degree 0

$R_{\mathrm{lens}} = R$

# Coperfection for the $q$-torus

$(A, I) =$ coperfect'n of $\left(\mathbb{Z}_p[[q^{p^{-\infty}}-1]], ([p]_q)\right)$

$q^{p^{-n}}$ $\qquad \frac{q^p-1}{q-1}$.

$$R = \bar{A}\langle x^{\pm 1}\rangle^\wedge_{(p)}$$

$$\bar{A} = \mathbb{Z}_p[\mu_{p^\infty}]^\wedge_{(p)}$$

In this case, $H^1(\Delta_{R/A_{perf}}), H^1(R_{coperf}) \neq 0$.

$$\Delta_{R/\bar{A}_{perf}} = \left(A\langle x^{\pm p^{-\infty}}\rangle \xrightarrow{\gamma - 1} J\, A\langle x^{\pm p^{-\infty}}\rangle^\wedge_{(p,I)}\right)$$

$J = \ker(A \to \mathbb{Z}_p)$

$q^{p^{-1}} \mapsto 1$

$$\gamma(x^i) = q^i x^i$$

$\Rightarrow$ in degree 1, $(q-1)\cdot 1$ is not a coboundary

<u>even</u> mod $[p]_q$.

– nontrivial $H^1$ relates to étale cohomology

# Base change compatibility

$(A, I)$ perfect

$R \mapsto R_{lens}$ on derived $p$-complete $\bar{A}$-algebras

commutes with faithfully flat base change on $(A, I)$

Ingredients: (assume $R$ bounded $p$-power torsion)

$(A, I) \to (B, IB)$ faithfully flat.

$$R_{lens} \hat{\otimes}^L_{\bar{A}} \bar{B} \simeq S_{lens} \qquad S = R \hat{\otimes}^L_{\bar{A}} \bar{B}$$

(in degree 0)

$$\|$$

$$\bar{\Delta}_{R/A} \hat{\otimes}^L_{\bar{A}} \bar{B} \simeq \bar{\Delta}_{S/B} \Leftarrow$$

H-T filtration

use analogous compatibility for cotangent complexes.

# Coconnectivity of coperfection

$(A, I)$ = perfect prism $\quad R$ = derived $p$-complete $\bar{A}$-alg.

Then $\Delta_{R/A,\mathrm{perf}} \in D^{\geq 0}_{\mathrm{comp}}(A)$.

$\Uparrow$

$\Delta_{R/A,\mathrm{perf}}/p \in D^{\geq 0}_{\mathrm{comp}}(A/p)$

$\Uparrow$

Lemma: $R_\bullet$ = simplicial ring over $\mathbb{F}_p$
then Frobenius acts by $0$ on $H^{-i}(R_\bullet)$ $\forall\, i > 0$.

Pf: reduce to an explicit calculation.
$i=1$: work explicitly on $S_1 = \mathbb{F}_p \otimes^L_{\mathbb{F}_p[x]} \mathbb{F}_p = \mathrm{Sym}_{\mathbb{F}_p} \mathbb{F}_p(1)$
(and similar for higher $i$)

# Coperfection for a semilens

(Reminder: if $R$ = semilens then $\mathbb{D}_{R/A,\text{perf}} \hookrightarrow D^{\le 0}_{\text{comp}}(A)$.)

In general, if $\mathbb{D}_{R/A,\text{perf}} \in D^{\le 0}_{\text{comp}}(A)$, then

— concentrated in degree 0, where it's a perfect $(p, I)$-complete $\delta$-rings.

— $(\mathbb{D}_{R/A,\text{perf}}, I\mathbb{D}_{R/A,\text{perf}})$ is a perfect prism over $(A, I)$

and $R_{\text{lens}}$ = concentrated in degree 0 where is a lens.

$R \to R_{\text{lens}}$ is universal map from $R$ to a lens.

$\Rightarrow$ $\{\text{semilenses}\} \subset \{\text{lenses}\}$ has a left adjoint.

# Adjunction of roots to a perfect prism

$(A, I)$ = perfect prism $\quad P \in \bar{A}[x]$ monic

$\Rightarrow \exists$ faithfully flat $(A, I) \to (B, IB)$

s.t. $P$ has a root in $\bar{B}$.

Pf $R = \bar{A}[x^{p^{-\infty}}]^{\wedge}_{p} / (P)$ is $p$-completely faithfully flat over $\bar{A}$.

is a regular semiperfectoid.

$B = \Delta_{R/A, \mathrm{perf}}$ works.

# The André flatness lemma

Let $R$ = lens

then $\exists$ [illegible] faithfully flat morph

$R \to S$ of lenses s.t.

$S$ is absolutely integrally closed

(every monic poly has a root).

# The direct summand conjecture

One ingredient in

**Thm** (André, Bhatt) (Conjecture of Hochster)

$R$ = regular local ring

$R \to S$ finite injective rings map

$\Rightarrow$ splits in $\underline{\mathrm{Mod}}_S$

also need perfectoid Abhyankar lemma

(a form of almost purity)

# Zariski closed vs. strongly Zariski closed

<u>Cor</u> $\{$ g sem, lens $\} \to \{$ lens $\}$

left adjoint.

infact, $R$ = sem, lens

$R \to R_{lens}$ is <u>surjective</u> !!