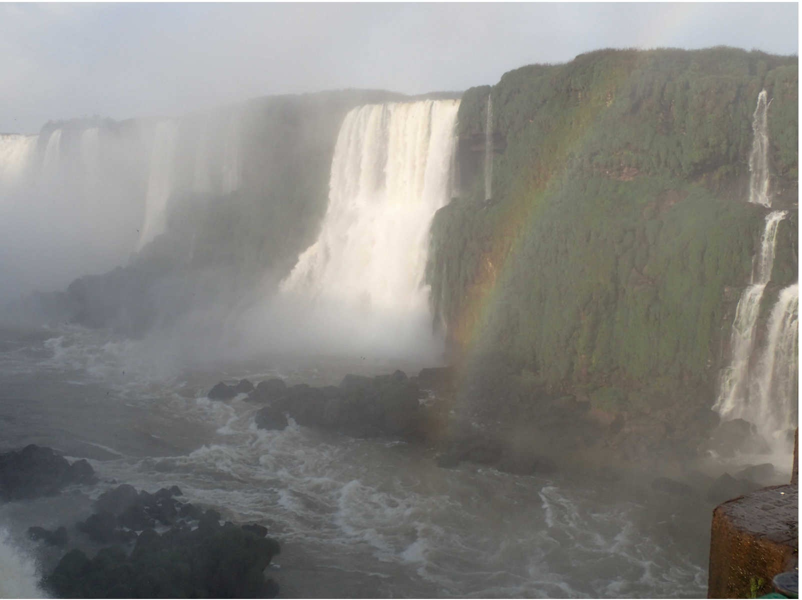
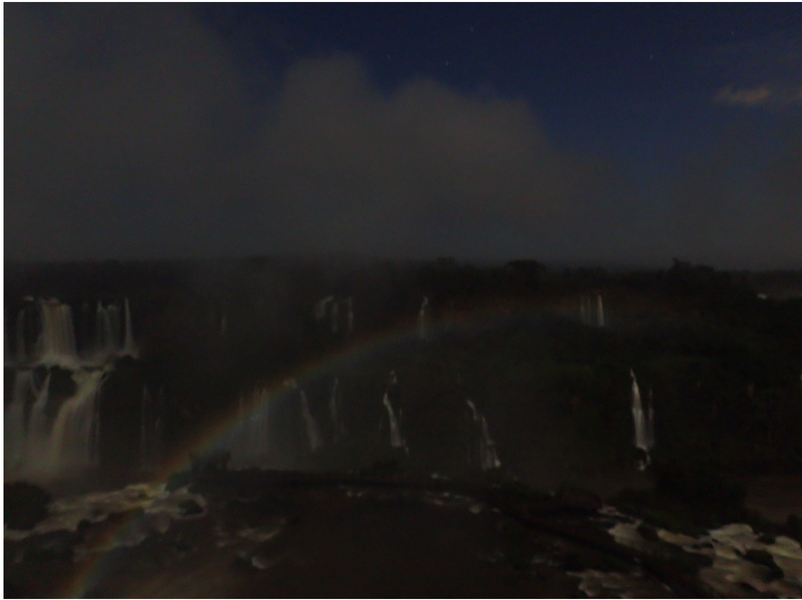
The arc-topology and friends

(h-topology, v-topology)

Iguaçu Falls by day and night, 2018





Carryover: lens coperfection of a semilens is a quotient

R = semilens (derived p-complete quotient of a lens)

$R \to R_{lens}$ universal map to a lens.

lens coperfection

<u>Thm</u> $R \to R_{lens}$ is <u>surjective</u>

<u>Pf</u> Check this after a p-completely flat cover change so I can assume $\bar{A}$ = abs. integrally closed (AIC)

and $\bar{A} \to R$ surjective. ($\Rightarrow$ #: $\bar{A}^b \to \bar{A}$ <u>surjective</u>)

$J = \ker(\bar{A} \to R)$ choose $x_i \in \bar{A}^b$ which # to generators of J

check: $\bar{A}/(x_i^{\# p^{-j}})^{\wedge}_{(p)}$ is a lens satisfies universal property of $R \to R_{lens}$, so it <u>is</u> R_{lens}.

Grothendieck topologies revisited

$\mathcal{C}$ = category (e.g. Sch or AffSch = Ring^{op})

Which families of morphisms $\{U_i \to U\}_{i \in I}$ form coverings? (tells you what sheaves are)

- any isomorphism is a covering
- $\{U_i \to U\}_{i \in I}$, $\{V_{ij} \to U_i\}_{j \in J_i}$ coverings
  $\Rightarrow \{V_{ij} \to U_i \to U\}_{i \in I, j \in J_i}$ is a covering
- $\{U_i \to U\}_{i \in I}$ covering, $V \to U$ any morphism
  $\Rightarrow U_i \times_U V$ exists $\forall i$ and $\{U_i \times_U V \to V\}_i$ is a covering.

$\leadsto$ site, topos

Valuation rings and arcs

Valuation ring = local integral domain V
maximal among subrings of Frac V for local inclusions
$= K$

$\Gamma = K^\times / V^\times$ is totally ordered with inclusion.
= value group $\quad v : K^\times \to \Gamma$ is the Krull valuation
associated to V.

A valuation ring V is eudoxian/microbial/rank 1
if Γ embeds into $(\mathbb{R}, +)$ (as opposed to $(\mathbb{R} \times \mathbb{R}, +)$ lexicographic)

An arc = Spec V $\quad V$ = eudoxian valuation ring

(a trait/dash is Spec DVR)

Eudoxian/microbial valuations

V = valuation ring
Γ = value group
(written additively)

TFAE:

- Γ embeds into $(\mathbb{R}, +)$
- For $\alpha, \beta \in \Gamma > 0$, $\exists$ positive integer n s.t. $n\alpha > \beta$
- Spec V has at most 2 points. (generic/special point)

Arc-coverings and v-coverings (Rydh)

$$\begin{tikzcd} \mathrm{Spec}(W) \arrow[r, dashed] \arrow[d, dashed] & Y \arrow[d, "f"] \\ \mathrm{Spec}(V) \arrow[r] & X \end{tikzcd}$$

f is a <u>v-covering</u> if $\forall$ diagrams at left with V a valuation ring, there exists a completion where $V \to W$ is faithfully flat morphism of valuation rings

f is an <u>arc-covering</u> if this holds whenever V is eudoxian (so $\mathrm{Spec}(V)$ is an arc) and W is also.

Examples and nonexamples

- any faithfully flat map is an v-coverings
  ⇒ arc-coverings
  (lift closed point first)

- f proper surjective, its an v-coverings
  ⇒ arc-coverings
  (lift generic point first!)

e.g. $X = \operatorname{Spec} R$ R noetherian

$$R \to \prod_{\mathfrak{m}} (R_{\mathfrak{m}})^{\wedge}$$ faithfully flat

The h-topology, the v-topology, and the arc-topology

h-topology = generated by étale coverings, ~~proper~~ proper surjective morphisms.
(Voevodsky)

($X_{red} \to X$)

(v)
arc-topology: $\{f_i : Y_i \to X\}_{i \in I}$ is an covering
if $\forall$ open affine $V \subseteq X$, $\exists\, t : K \to I$ finite and
affine opens $U_K \subseteq f_{t(K)}^{-1}(V)$ s.t. $\coprod U_K \to V$.
is an arc-covering
(v)

A construction of v-coverings

A = ring $(A \to V_i)_{i \in I}$ representative maps from A to AIC-valuation rings of cardinality at most $\max(\aleph_0, \# A)$

then $A \to \prod_{i \in I} V_i$ is a v-covering.

(given $A \to V$, replace V with smallest valuation subring containing image of V, then pass to absolute integral closure)

e.g. $\mathbb{Z} \to \prod_p \mathcal{O}_{\overline{\mathbb{Q}}_p}$.

An arc-covering which is not a v-covering

V = valuation ring <u>not</u> eudoxian

$\mathfrak{p} \in \mathrm{Spec}(V)$ not generic or closed point.

$V \longrightarrow V_{\mathfrak{p}} \times V/\mathfrak{p}$ is an arc-covering but not a v-covering.

Statement: arc-descent for perfect schemes

$A \to B$ arc-covering of perfect $\mathbb{F}_p$-algebras.

Then augmented $\check{C}$-A complex

$$0 \to A \to B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \cdots$$

is exact. (and similarly for schemes)

- can define a vector bundle on Spec(A) using descent data for Spec(B) $\to$ Spec A

Statement: arc-descent for lenses

Next time:

similar for lenses

using ~~ar~~ arc $\mathcal{P}$-topology

Statement: arc-descent for étale cohomology

$R \in \underline{\text{Ring}}$

$\mathcal{F}$ = torsion sheaf on $(\operatorname{Spec} R)_{et}$

then $f : (X \to \operatorname{Spec} R) \mapsto R\Gamma(X_{et}, f^* \mathcal{F})$

satisfies descent for arc-topology