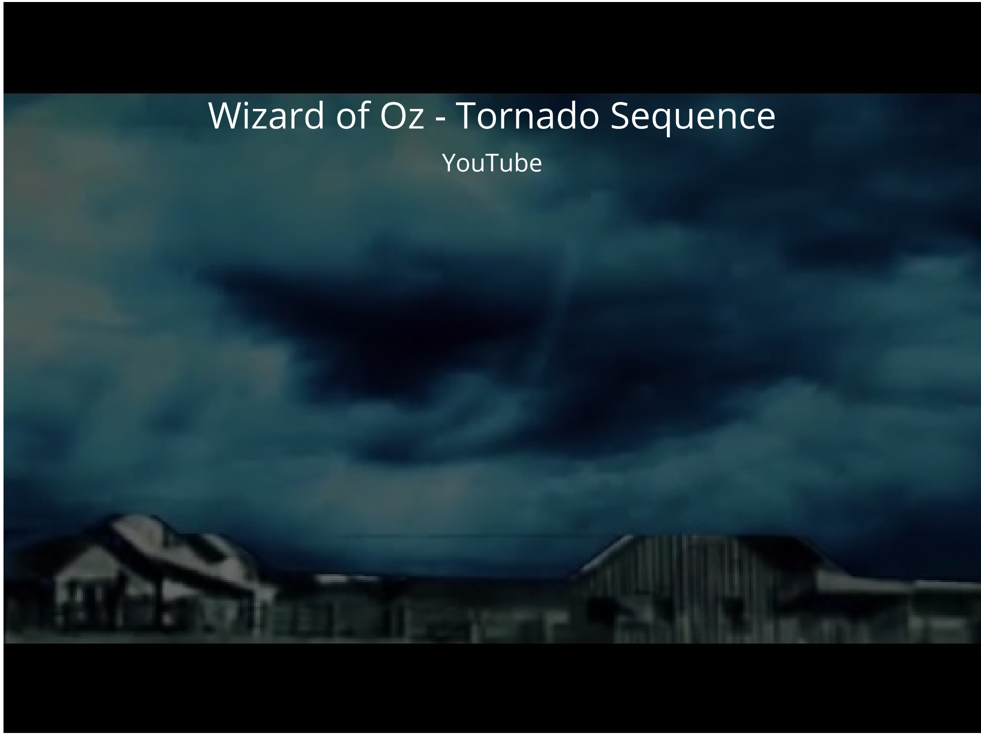
Arc-descent

has height 1 prime
rank 1

Correction from last time: "microbial" is not a synonym for "eudoxian". Eudoxian valuations are microbial but not *vice versa*.

And a clarification: when you fill in a diagram as below to verify that the right vertical arrow is an arc-covering, the map V -> W should be faithfully flat, but *need not* be integral.

$$\begin{tikzcd}
\mathrm{Spec}(W) \arrow[r, dashed, "\text{arc}"] \arrow[d, dashed, "\text{f.flat}"] & Y \arrow[d] \\
\mathrm{Spec}(V) \arrow[r, "\text{arc}"] & X
\end{tikzcd}$$



Pullback of perfect schemes

(all perfect $\mathbb{F}_p$-schemes)

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

$K^\bullet$ = any bdd quasicoherent sheaves on X

Then $Lg^* Rf_* K^\bullet \longrightarrow Rf'_* Lg'^* K^\bullet$

is a quasi-isom.

__Pf__ reduce to all affine,

follows from: $A \to B$, $A \to C$ are morphisms of perfect $\mathbb{F}_p$-algebras

then $\mathrm{Tor}_i^A(B, C) = 0$ for all $i > 0$.

(exercise: reduce to case $A \to B$ is quotient by $\sqrt{(f)}$

check directly)

Descent for a perfect blowup: statement

(Bhatt - Scholze - Takagi)

Vect(-) = category of vector bundles (as a groupoid)

Lemma X = noetherian $\mathbb{F}_p$-scheme

Z = closed subscheme

$f : Y \to X$ a blowup centered within Z

$E = f^{-1}(Z)$ perfection of scheme

1) Given $\mathcal{F} \in \mathrm{Vect}(X_{perf})$, then this Δ is distinguished.

$$R\Gamma_*(X_{perf}, \mathcal{F}) \to R\Gamma_*(Y_{perf}, \mathcal{F}) \oplus R\Gamma_*(Z_{perf}, \mathcal{F}) \to R\Gamma_*(E_{perf}, \mathcal{F}) \to$$

$*$ = Zariski, fppf, Nisnevich, ...

étale

2) $\mathrm{Vect}(X_{perf}) \to \mathrm{Vect}(Y_{perf}) \times_{\mathrm{Vect}(E_{perf})} \mathrm{Vect}(Z_{perf})$

is an equivalence of groupoids

Descent for a perfect blowup: acyclicity

$X = \operatorname{Spec} A$

$Z = \operatorname{Spec} A/I$

$nE = V(I^n)$ on Y.

1) $\mathcal{F} = \mathcal{O}$.

$$0 \to \mathcal{O}(X_{pf}) \to \mathcal{O}(Y_{pf}) \oplus \mathcal{O}(Z_{pf}) \to \mathcal{O}(E_{pf}) \to 0 \quad \text{exact, } A\text{ str.}$$

$\cong$ $\cong$

Need: $H^i(Y_{pf}, \mathcal{O}) \xrightarrow{\cong} H^i(E_{pf}, \mathcal{O})$ isom $\forall i > 0$.

(formal functions thm)

$\exists\, c$ s.t. $\operatorname{Ker}(H^i(Y, \mathcal{O}) \to H^i(E_n, \mathcal{O})) \subseteq I^{n-c} H^i(Y, \mathcal{O})$

for $n \gg 0$

$\operatorname{Im}(H^i(E_m, \mathcal{O}) \to H^i(E_n, \mathcal{O})) = \operatorname{Im}(H^i(Y, \mathcal{O}) \to H^i(E_n, \mathcal{O}))$

and for $m \gg n \gg 0$.

$\varphi^e : H^i(E_n, \mathcal{O}) \to H^i(E_n, \mathcal{O})$ has image contained in $\operatorname{Im}(H^i(Y, \mathcal{O}))$

$\varphi^e \searrow H^i(E_{npe}, \mathcal{O}) \nearrow$

$$H^i(Y_{perf}, \mathcal{O}) = \operatorname{colim}_{\varphi} H^i(E_n, \mathcal{O}) \longrightarrow \operatorname{colim}_{\varphi} H^i(E, \mathcal{O}) = H^i(E_{perf}, \mathcal{O})$$

for all $n \gg 0$

Descent for a perfect blowup: glueing for vector bundles

Assume now A = I-adically complete (by Beauville–Laszlo)

start with $\mathrm{Vect}(Y) \times_{\mathrm{Vect}(E)} \mathrm{Vect}(Z)$ ← these lift to X

Pick n s.t. $H^1(Y, I^k/I^{k+1}) = 0$ $(\forall k \geq n)$.

Given $\xi \in \mathrm{Vect}(X)$, $\eta \in \mathrm{Vect}(Y)$, isom along kE.

claim can lift this isom to $(k+1)E$.

Obstruction to lifting lives in

$$H^1(Y, I^k/I^{k+1} \otimes \mathcal{H}om(f^*\xi, \eta)).$$

$= 0$

Arc-descent for perfect schemes

Conclusion:

- structure presheaf on perfect $\mathbb{F}_p$-schemes an arc-sheaf.
  and for X affine, higher cohom vanishes.

- $\underline{\text{Vect}}$ on this category is an arc-stack.

e.g. $A \to B$ arc-covering of perfect $\mathbb{F}_p$-algebras

$$0 \to A \to B \to B \otimes_A B \to B \otimes_A B \otimes_A B \to \cdots$$

is acyclic.

An auxiliary glueing lemma

For arc-topology, I also need to deal with $\mathrm{Spec}(V) \to V_f \oplus V/f)$

$$\begin{array}{ccc} \mathbf{Vect}(R) & \longrightarrow & \mathbf{Vect}(R_1) \\ \downarrow & & \downarrow \\ \mathbf{Vect}(R_2) & \longrightarrow & \mathbf{Vect}(R_{12}) \end{array}$$

Lemma: $\begin{array}{ccc} R & \longrightarrow & R_1 = R[s^{-1}] \\ \downarrow & & \downarrow \\ R_2 & \longrightarrow & R_{12} \end{array}$ $\to R_2[s^{-1}]$

s.t. $0 \to R \to R_1 \oplus R_2 \to R_{12} \to 0$ exact.

(i.e. $R_1/R = R_{12}/R_2$)

Then this square is cartesian

e.g. $R_1 = R[f^{-1}]$ for not zero div..

$R_2 = \hat{R}_{(f)}$ $\Rightarrow$ Beauville–Laszlo theorem.

Comments on the proof of the glueing lemma

M_1, M_2, M_{12} fin proj modules

$M := \ker(M_1 \oplus M_2 \to M_{12})$

Claim: $M \otimes_R R_i \to M_i$ is surjective

$M \otimes_R R_i \to M_i$ injective

$$\begin{tikzcd}
 & N \otimes_R R_i \arrow[r] \arrow[d] & F_i \arrow[r] \arrow[d, equal] & M \otimes_R R_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N_i \arrow[r] & F_i \arrow[r] & M_i \arrow[r] & 0
\end{tikzcd}$$

M projective.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & N_1 \oplus N_2 \arrow[r] \arrow[d] & N_{12} \arrow[r, dashed] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d] & F_1 \oplus F_2 \arrow[r] \arrow[d] & F_{12} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d, dashed] & M_1 \oplus M_2 \arrow[r] \arrow[d] & M_{12} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \star \arrow[r] \arrow[d] & \mathrm{Hom}_{R_1}(M_1, N_1) \oplus \mathrm{Hom}_{R_1}(M_2, N_2) \arrow[r] \arrow[d] & \mathrm{Hom}_{R_{12}}(M_{12}, N_{12}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \star \arrow[r] \arrow[d] & \mathrm{Hom}_{R_1}(M_1, F_1) \oplus \mathrm{Hom}_{R_1}(M_2, F_2) \arrow[r] \arrow[d] & \mathrm{Hom}_{R_{12}}(M_{12}, F_{12}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \star \arrow[r] \arrow[d] & \mathrm{Hom}_{R_1}(M_1, M_1) \oplus \mathrm{Hom}_{R_1}(M_2, M_2) \arrow[r] \arrow[d] & \mathrm{Hom}_{R_{12}}(M_{12}, M_{12}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Arc-descent for étale cohomology

$R \in \underline{\text{Rings}}$ $\quad \mathcal{F}$ torsion sheaf on $(\operatorname{Spec} R)_{et}$

$f : X \to \operatorname{Spec} R$

$\underline{R\Gamma(X_{et}, f^*\mathcal{F})} = R\Gamma(X_{arc}, f^*\mathcal{F})$

— $f : Y \to X$ arc-covering

<u>Cases:</u> $\quad$ f faithfully flat ✓

f proper surjective

reduce to $X = \operatorname{Spec}$ (strictly henselian local rings)

$\longrightarrow$ reduce to $X = \underline{\text{point}}$ using proper base change thm.

$\to$ ~~map~~ $Y \to X$ has a section,

Arc-descent for étale cohomology

$$\mathrm{Spec}\left(V \longrightarrow V_{\mathfrak{p}} \oplus V/\mathfrak{p} \right)$$

where V is AIC

$\Rightarrow$ $V/\mathfrak{p}$ is also AIC

$\Rightarrow$

$$R\Gamma(V) \longrightarrow R\Gamma(V_{\mathfrak{p}}) \oplus R\Gamma(V/\mathfrak{p}) \to R\Gamma(\kappa(\mathfrak{p}))$$