The étale comparison theorem

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The Artin-Schreier-Witt exact sequence

Let $X \in \underline{Sch}_{\mathbb{F}_p}$. On $X_{ét}$, the sequence

$$0 \to \underline{\mathbb{F}_p} \to \mathbb{G}_a \xrightarrow{\varphi - 1} \mathbb{G}_a \to 0$$

($R \to R[x]/(x^p - x - y)$ is finite étale for any $y \in R$)

similarly is exact.

$$0 \to \underline{\mathbb{Z}/p^n\mathbb{Z}} \to W_n \xrightarrow{\varphi - 1} W_n \to 0$$

i.e. $\underline{\mathbb{Z}/p^n\mathbb{Z}} = \mathrm{Coker}(W_n \xrightarrow{\varphi - 1} W_n) = W_n^{\varphi = 1}$

(compare to $1 \to \mu_p \to \mathbb{G}_m \xrightarrow{x \mapsto x^p} \mathbb{G}_m \to 1$)

Statement of the étale comparison theorem

Let (A, I) be a perfect prism $\quad I = (d)$

R = derived p-complete $\bar{A}$-algebra.

For $n \geq 1$, $\exists$ a canonical isom

$$\to R\Gamma_{et}(\operatorname{Spec} R(p^{-1}), \underline{\mathbb{Z}/p^n\mathbb{Z}}) \cong \left(\Delta_{R/A}[d^{-1}]/p^n\right)^{\phi=1}$$

$$R\Gamma_{et}(\operatorname{Spec} R, \underline{\mathbb{Z}/p^n\mathbb{Z}}) \cong \left(\Delta_{R/A}/p^n\right)^{\phi=1}$$

Frobenius fixed points and truncation

"derived invariants"

$K \in D(\sim)$ $\quad K \circlearrowleft \varphi$ $\quad K^{\varphi=1} = \mathrm{core}(K \to K)$

<u>Lemma</u> $B \in \mathrm{Ring}_{\mathbb{F}_p}$ $\quad t \in B$ $\quad D_{comp}(B)$

derived t-complete derived cat.

$N \in D_{comp}(B)$ $\quad \varphi : N \to N \Rightarrow$ $\boxed{N^{\varphi=1} \simeq (N/t)^{\varphi=1}}$

($F = \mathrm{core}(N \to N/t) \Rightarrow F$ is t-complete and φ-action is topologically nilpotent.

so $(1-\varphi)^{-1} = 1 + \varphi + \varphi^2 + \cdots$

Frobenius fixed points and coperfection

Cor: $D_{comp}(B(F)) \longrightarrow D(\mathbb{F}_p)$ $\quad M \to M^{\varphi=1}$

the functor $\quad M \to M(t^{-1})^{\varphi=1}$

$= D_{comp}(B) + \varphi$-action

Go then commute with colimits.

(i.e. commute with completing the colimit ordinary)

$\Rightarrow$ In notation of étale comparison theorem,

$(\Delta_{R/A}/p^n)^{\varphi=1} \xrightarrow{\sim} (\Delta_{R/A,perf}/p^n)^{\varphi=1}$

$(\Delta_{R/A}[A^{-1}]/p^n)^{\varphi=1} \cong (\Delta_{R/A,perf}[d^{-1}]/p^n)^{\varphi=1}$

(reduce to $n=1$)

The arc_p-topology and the arc-topology

$f: R \to S$ a morphism of derived p-complete rings

f is an arc$_p$-covering if diagram-filling condition holds whenever V is a p-complete rank ≤ 1 valuation ring

$$\begin{tikzcd} \operatorname{Spec}(W) \arrow[r, dashed] \arrow[d, "i"'] & \operatorname{Spec} S \arrow[d] \\ \operatorname{Spec}(V) \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

i faithfully flat

Lemma If $f: R \to S$ is an arc$_p$-covering then $R \to S \times R/p \times R[p^{-1}]$ is an arc-covering.

pf $R \to V$ image of p is:
- zero
- unit
- neither (pass to p-completion)

Arc p-descent for étale cohomology

$R \in \underline{\text{Rings}}$, $\mathcal{F}$ = torsion sheaf on $(\operatorname{Spec} R)_{et}$,

then $(f : \operatorname{Spec} S \to \operatorname{Spec} R) \mapsto R\Gamma_{et}(\operatorname{Spec} \hat{S}_{(p)}[p^{-1}], \text{pullback of } \mathcal{F})$

satisfies descent for arc$_p$-topology.

(use previous lemma to reduce to arc-descent

note: derived p-completion $+ (p^{-1})$ kills both complexes of (R/p)-modules and $\mathbb{Z}(p^{-1})$-modules

Lenses as valuation rings

Lemma V = p-complete AIC valuation ring
then V is a lens. (Easy)

Lemma V = lens. Then V is a valuation ring
$\Leftrightarrow$ $V^\flat$ is a valuation ring.

if so, same value group & same residue field.

Pf $\#: V^\flat \to V$ multiplicative $x^\# = \theta([x])$
gives a map $\{$principal fract. ideals of $V^\flat\} \to \{$princip. frac. ideals of $V\}$
preserving the orderings.

so V valuation ring $\Rightarrow V^\flat$ is too.

Lenses as valuation rings

$\varpi \in V$ in image of $\sharp$
s.t. ker(Frob on $V/(p)$) gen by $\varpi^{1/p}$. $\cong V^\flat / \sim$

If $V^\flat$ is a valuation ring.

$x, y \in V$ s.t. not both zero mod ϖ.

WLOG $x = yz + pu$ & then argue that $y | \varpi | p$.

$\varpi = y w + p v$

$(1 - p/\varpi)$ = unit.

$\Rightarrow V$ valuation ring.

Similarly:

V = lens which is an A/K valuation ring

Then so is $V^\flat$.

(untilt any finite extension of $V^\flat$ to extension of V.)

(Converse is true but deeper.)

Arc$_p$-descent for lenses

Lemma: $R \to S$ are p-coverings of lenses

then $R^b \to S^b \oplus R^b[d^{-1}]$

is arc-covering.

($R^b \to V$ ← either d maps to a unit
or can take completion,
then unit.)

Cor: $0 \to R \to S \to S \hat{\otimes}_R S \to \cdots$

is acyclic

(go through $R^b \to W(R^b) \to R$).

The comparison theorem: application of arc_p-descent

Want to show

$$R\Gamma_{et}(\mathrm{Spec}\, R[p^{-1}], \mathbb{Z}/p^n\mathbb{Z}) \simeq \left(\Delta_{R/A,\mathrm{perf}}[d^{-1}]/p^n\right)^{\varphi=1}$$

Both sides satisfy arc_p-descent.

Can reduce to $R = \prod V_i$ product of p-complete valuation rings.

The comparison theorem: an arc p-local calculation

For $R = V_i$, set from A-S-W

$$0 \to \mathbb{Z}/p^n\mathbb{Z} \to W(R^\flat)/p^n \xrightarrow{\varphi - 1} W(R^\flat)/p^n \to 0$$

(in general)

$$W(R^\flat) \cong \Delta_{R/A, \mathrm{perf}}$$

$$0 \to \mathbb{Z}/p^n\mathbb{Z} \to W(R^\flat)(d^{-1})/p^n \to \frac{W(R^\flat)(d^{-1})}{p^n} \to 0.$$

$\Longrightarrow$ both maps a isomorphism.