Applications of the étale comparison theorem

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Will the Metallica table help pinball "ride the lightning" to relevance?

It wasn't that long ago that pinball seemed like it might be heading for that big drain in the sky. Even with the stealth rebirth of the arcade generating new interest, good luck finding a pinball machine instead of a redemption game that spits out tick...

Reminder: statement of the étale comparison theorem

(A, I) perfect prism $\qquad I = (d)$

R = derived p-complete $\bar{A}$-algebra

$$R\Gamma_{et}\left(\mathrm{Spec}\, R[p^{-1}], \underline{\mathbb{Z}/p^n\mathbb{Z}}\right) \simeq \Big(\underbrace{\Delta_{R/A}[d^{-1}]/p^n}\Big)^{\varphi=1}$$

$$\simeq \left(\Delta_{R/A,\mathrm{perf}}[d^{-1}]/p^n\right)^{\varphi=1}$$

Tilting of valuation rings revisited

Thm $V = $ Lens. Then V is an AIC valuation ring iff $V^\flat$ is.

Need to prove "if" direction — divisible, alg closed

Argument: V, $V^\flat$ have same value group, residue field

$\Rightarrow$ Frac V can't have any finite unramified extension or any finite tamely ramified extension.

If Frac V is not alg. closed, then any nontrivial finite Galois ext is totally wildly ramified, so its Galois group is a p-group & has a $\mathbb{Z}/p\mathbb{Z}$-quot $\Rightarrow H^1_{et}(\mathrm{Spec}\, V[p^{-1}], \mathbb{Z}/p\mathbb{Z}) \neq 0$

$\Rightarrow\!\!\!\!\!\not\,$ by etale comparison

How this relates to perfectoid fields

<u>Thm</u> K = perfectoid field. $K^\flat$ ~~=~~ its tilt

then $G_K \cong G_{K^\flat}$. uses Krasner's lemma

in particular, $K = \overline{K} \iff K^\flat = \overline{K^\flat}$ ⟸ previous slide

Hard part: show that finite extension of a perfectoid field is still a perfectoid field.

Here we use something easier:

K embeds into a perfectoid field which is AC.

(take $\widehat{\overline{K}}$)

A lemma on Frobenius invariants

k = algebraically closed field of char $p > 0$

$M \in D(k)$ perfect complex (i.e. up to quasi-isom, bounded, finite dim)

$\circlearrowleft$
φ. Then $\forall i$, $H^i(M^{\varphi=1}) \otimes_{\mathbb{F}_p} k \to H^i(M)$

is injective, with equality iff $\varphi : H^i(M)$ is bijective.

Key case: if $M = M[0]$ and φ bijective, then

$\underbrace{M^{\varphi=1}} \otimes_{\mathbb{F}_p} k \cong M$ (nonabelian Artin-Schreier, Lang, Katz

$0 \to \underline{\mathbb{F}_p} \to M \xrightarrow{\varphi - 1} M \to 0$

Dimensions in étale and de Rham cohomology

C = complete, alg closed extension of $\mathbb{Q}_p$ valring $\mathcal{O}_C$, residue field k

$\operatorname{Spec} \mathcal{O}_C = \{\eta, \operatorname{Spec} k\}$

X = smooth proper $\mathcal{O}_C$-scheme

X_η = generic fibre, X_k = special fibre.

Thm. $\forall i \geq 0$,

$$\dim_{\mathbb{F}_p} H^i_{et}(X_\eta, \mathbb{F}_p) \leq \dim_k \underline{H^i_{dR}(X_k/k)}$$

Pf sketch: $\mathcal{O}_C = A/I$ $\quad (A, I) \to (W, \varphi)$!

$R\Gamma_A(X) = R\Gamma(X, \Delta_{X/A})$ $\quad$ coincide with $H^i(R\Gamma_A(X) \otimes^{\mathbb{L}}_A k)$

Tate twists

uses crystalline comparison

$$\dim_{\mathbb{F}_p} H^i_{et}(X_\eta, \mathbb{F}_p) \le \underbrace{\dim_k H^i_{dR}(X_k/k)}$$

$\dim_{\mathbb{C}^\flat}$ $\|$ (uses previous lemma + étale comp.) $\dim_k$ $\|$

$$H^i(R\Gamma_A(X) \otimes^L_A \mathbb{C}^\flat) \le H^i(R\Gamma_A(X) \otimes^L_A k)$$

$\leftarrow$ semicontinuity for perfect complexes over $\mathcal{O}_{\mathbb{C}^\flat}$.

Lemma: $K^\bullet$ = perfect complex in $D(\mathcal{O}_{\mathbb{C}^\flat})$

For each i,

$$\dim_{\mathbb{C}^\flat} H^i(K^\bullet \otimes^L_{\mathcal{O}_{C^\flat}} \mathbb{C}^\flat) \le \dim_k H^i(K^\bullet \otimes^L_{\mathcal{O}_{\mathbb{C}^\flat}} k)$$

e.g. $\mathcal{O}_{C^\flat} \xrightarrow{\varpi} \mathcal{O}_{C^\flat}$

Comparison of the two constructions

$\mu_{p^m} = \ker(\mathbb{G}_m \xrightarrow{p^m} \mathbb{G}_m)$

on X_{arc} $\quad \mathbb{Z}_p(1) = \varprojlim \mu_{p^m}$

$\mathbb{Z}_p(n) = \mathbb{Z}_p(1)^{\otimes n}$ tens (over $\mathbb{Z}_p$)

<u>Lemma</u> R I-tors. fr. / $n > 0$

$$R\Gamma_{et}(\operatorname{Spec} R[\tfrac{1}{p}], \mathbb{Z}_p(n)) = \left(\varphi^{-1}(d)^n A \xrightarrow{\frac{\varphi}{d^{n-1}}} A\right)$$

$R = (A/I)$ $\quad I = (d)$

for $n = 0$

$$R\Gamma_{et}(\operatorname{Spec} R, \underbrace{\mathbb{Z}_p}_{= \mathbb{Z}_p(0)}) = \left(A_{(0)} \xrightarrow{\varphi - 1} A\right)$$

This gives a prismatic interpretation of étale cohom with $\underline{\mu_{p^n}}$-coefficients

(again, we use arc-descent to reduce to a calculation)

Tate twists and the comparison theorem

Preview: F_p-cohomological dimension

The key lemma and what it needs