Almost purity

Purity of the branch locus and Abhyankar's lemma

Thm (Zariski-Nagata purity) $\quad Z(xy-z^2)$

$X \hookrightarrow \bar{X}$ open immersion of regular noetherian schemes s.t. $\operatorname{codim}(\bar{X}-X, \bar{X}) \geq 2$.

Then every finite étale covering of X extends uniquely to $\bar{X}$

Abhyankar's lemma: X regular, D = snc divisor on X

$U = X - D \quad f: V \to U$ finite étale, tamely ramified along D.

then étale locally on X, f is simply adjoining roots of parameters cutting out components of D

Contexts for almost commutative algebra

Context: pair $(V, \underline{m})$ s.t. $\underline{m}^2 = \underline{m}$
ring, ideal

e.g. maximal ideal in a nondiscrete valuation ring

(aside: sometimes assume $\underline{m} \otimes_V \underline{m}$ is flat.)

e.g. $(\mathbb{Z}, (1))$ classical limit

Almost zero modules

$(V, \mathfrak{m})$ = context

$M \in \underline{\mathrm{Mod}}_V$ is <u>almost zero</u> if $\mathfrak{m}M = 0$.

These form a <u>thick</u> Serre subcategory of $\underline{\mathrm{Mod}}_V$

e.g. $0 \to M_1 \to M \to M_2 \to 0$ with M_1, M_2 almost zero

then M is almost zero.

Can formally quotient by almost zero modules

(i.e. formally invert $0 \to M$ when M is almost zero).

get category of <u>almost V-modules</u>.

e.g. $M \longrightarrow M_* = \mathrm{Hom}_V(\mathfrak{m}, M)$ is an <u>almost isomorphism</u>

module of almost elements

Almost finitely generated/projective modules

$(V, \mathfrak{m})$ context. $A \in \underline{\mathrm{Rings}}_V$

$M \in \underline{\mathrm{Mod}}_A$ is <u>almost finitely generated</u> if $\forall$ finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ $\exists$ f.g. A-submod $M_0 \subseteq M$ s.t. $\mathfrak{m}_0 M \subseteq M_0$

M is <u>almost projective</u> if the functor on $\underline{\mathrm{Mod}}_A$

$$N \longmapsto \mathrm{Hom}_A(M, N)_*$$ is <u>exact</u>

Almost finite étale morphisms

Fix context $(V, \underline{m})$

$f : A \to B$ morphism in $\underline{\mathrm{Rings}}_V$

f is <u>almost finite étale</u> if

B is almost finite projective A-module
(= almost finitely generated & almost projective)

<u>and</u> B is almost finite projective $B \otimes_A B$-module
via multiplication map.

(in classical limit, this agrees with the usual definition.)

Almost zero modules over a lens

R = lens J ideal of R

$J_{lens} = \ker(R \to (R/J)_{lens})$ this is surjective.

iso $(R/J)_{lens} = R/J_{lens}$

(e.g., $R = \mathbb{Z}_p[x^{p^{-\infty}}]^\wedge_{(p)}$ $J = (x)$, $J_{lens} = (x^{p^{-\infty}})^\wedge_{(p)}$

M = derived p-complete R module

M is J-almost zero if $J_{lens} M = 0$.

e.g. $J = (p)$
$J_{lens} = \sqrt{pR}$

Lemma: $J_{lens} \hat{\otimes}^L_R J_{lens} \cong J_{lens}$
$R/J_{lens} \hat{\otimes}^L_R R/J_{lens} \cong R/J_{lens}$ in $D_{comp}(R)$.
$J_{lens} \hat{\otimes}^L_R R/J_{lens} \cong 0$

The category of almost zero *p*-complete modules

category of J-almost zero derived p-complete R-complexes behaves well, $\cong$ cat of derived p-complete (R/J_{perf})-complexes.

Lem $K^\bullet \in D_{comp}(R)$ is J-almost zero $\iff J_{\mathrm{perf}} \hat{\otimes}^L_R K^\bullet \cong 0$.

Cor $K^\bullet \in D_{comp}(R)$ belongs to $D^{\leq 0}_{comp}(R)$ iff:

- $H^i(K^\bullet)$ is J-almost zero for $i > 0$
- $K^\bullet \hat{\otimes}^L_R R/J_{\mathrm{perf}} \in D^{\leq 0}_{comp}(R/J_{\mathrm{perf}})$.

Almost Galois extensions

($J_{\text{lens},n}$ = image of J_{lens} in R/p^n
then $(R/p^n, \bar{J}_{\text{lens},n})$ is a context.)

(V, m) context $\quad A \to B$ in $\underline{\text{Ring}}_V$.

G = finite group acting A-linearly on B.

$A \to B$ is $\underline{\text{almost } G\text{-Galois extension}}$ if

$A \to B^G$ almost isom.

and $\quad B \otimes_A B \longrightarrow \prod_{g \in G} B \qquad b \otimes b' \mapsto (\gamma(b)\, b')_{\gamma \in G}$

is an almost isomorphism.

($\Rightarrow$ almost finite étale)