Almost purity continued

Schedule adjustment: no office hours on Thursday, May 27. To make up for it, I'll have an extra office hour on Thursday, June 10. (Lectures and after-class office hours end Friday, June 4.)

Also, no lecture or office hours on Monday, May 31 (university holiday).

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Animals can do 'almost math'

When Christian Agrillo runs number-related experiments in his lab, he wishes his undergraduate subjects good luck. For certain tests, that's about all he says. Giving instructions to the people would be unfair to the fish. Yes, fish. Agrillo works at th...

Where this is going: integral extensions of lenses

R = lens $\quad R \to S$ finitely presented module-finite morphism

In general, S is not a lens. What can I say about S_{lens}?

- e.g. $R[x]/(x^2)$
- e.g. $\underbrace{\mathbb{Z}_p[x^{p^{-\infty}}, y^{p^{-\infty}}]^\wedge_{(p)}}_{R} / (x-y) \twoheadrightarrow S_{lens}$ (also divide by $x^{p^{-n}} - y^{p^{-n}}$)

e.g. $\underbrace{\mathbb{Z}_p[x^{p^{-\infty}}]^\wedge_{(p)}}_{(1?2)}[x^{\frac{1}{2}}] \longrightarrow S_{lens}$

$x^{\frac{1}{p}+\frac{1}{2}} \qquad x^{\frac{1}{2p^n}}$

Say $R[f^{-1}] \to S[f^{-1}]$ is finite étale. Try $J = (f)$ and well $\sqrt{J}$-almost context (or p-almost context)

Summary of almost commutative algebra

$(V, \mathfrak{m})$ context $\mathfrak{m}^2 = \mathfrak{m}$ (e.g. $\mathfrak{m} = (p^{1/p^\infty})$)

v.r.s, ideal

$M_* = \mathrm{Hom}_V(\mathfrak{m}, M)$

defined: almost zero modules ($\Rightarrow$ almost isomorphisms)

- almost finite projective modules
- almost finite étale rings maps

(A $\to$ B almost finite étale

$\Leftrightarrow$ B is almost finite projective as A-module and

as $B \otimes_A B$-module)

Almost Galois extensions are almost finite étale

$(V, \mathfrak{m})$ = context $A \to B$ in $\underline{\mathrm{Bang}}_V$, G = finite group acting A-linearly on B,

is almost G-Galois extension if

$A \to B^G$ almost isom

$B \otimes_A B \to \prod_{s \in G} B \quad b \otimes b' \mapsto (\gamma(b)b')_{\gamma \in G}$

is ~~an~~ almost isom.

This implies: 1) $A \to B$ almost finite étale

(need B almost fin proj over A. Use $\mathrm{tr}_{B/A} : B_* \to A_*$ $\quad \mathrm{tr}(b) = \sum_{s \in G} s(b)$

$(\eta \in \mathfrak{m})$

and elements $e_\eta \in B \otimes_A B$ $\quad e_\eta^2 = \eta e_\eta$ split off $\ker(\mu) \subseteq B \otimes_A B$.

2) $A \to C \to B$ where H = subgroup of G, ($C \to B^H$ almost isom)

$C \to B$ is almost H-Galois. $\Rightarrow A \to C$ almost finite étale.

Galois closures for rings

(no almost context)

$R \to S$ finite étale morphism of constant rank r

$\exists$ an S_r-Galois extension $R \to T$

factoring through an S_{r-1}-Galois ext $S \to T$

<u>Pf</u> $\operatorname{Spec} T$ = closed-open subscheme of

$\operatorname{Spec} S \times_{\operatorname{Spec} R} \operatorname{Spec} S \times \cdots$ (r times)

obtained by removing all partial diagonals.

Arc_p-descent and a pullback construction

J = ideal of S

Let $S \to S'$ integral morphism of derived p-complete rings.

Suppose: for every p-complete valuation rings V, every map $S \to V$ which does not kill J extends uniquely to S'.

$$\begin{tikzcd} S_{\mathrm{lens}} \arrow[r] \arrow[d] & S'_{\mathrm{lens}} \arrow[d] \\ (S/J)_{\mathrm{lens}} \arrow[r] & (S'/J)_{\mathrm{lens}} \end{tikzcd}$$

Then the square is a pullback in $D_{comp}(S)$ (Arc$_p$-descent for lenses)

(note: $S \longrightarrow S' \oplus S/J$ is an arc$_p$-covering, and $S' \hat{\otimes}_S S' \longrightarrow (S'/J \hat{\otimes}_{S/J} S'/J) \times S'$ is an arc$_p$-covering.

Cor: If $\operatorname{Spec} S' \to \operatorname{Spec} S$ is an isom outside $V(J)$ then $S_{\mathrm{lens}} \simeq S'_{\mathrm{lens}}$.

Almost purity (almost)

R = lens J = fin gen ideal of R

S = finitely presented, module-finite R-algebra

s.t. $\operatorname{Spec} S \to \operatorname{Spec} R$ is finite étale away from $V(J)$.

1) $J_{lens} S\, H^i(S_{lens}) = 0$ $(i > 0)$. ← improve this later.

2) $S \to S_{lens}$ is an isom away from $V(J)$

3) For $n > 0$, $R/p^n \to H^0(S_{lens})/p^n$ almost finite étale for context $(R/p^n, J_{lens,n})$ improve this later.

4) If $\operatorname{Spec} S \to \operatorname{Spec} R$ is a G-Galois cover outside $V(J)$ ~~then~~ $R \to H^0(S_{lens})$ is a J-almost G-Galois ext.

and this is a ring.

Proof of almost purity (almost)

André flatness!

step 4: - reduce to case where R is AIC in some sense, can find $f_1 \dots f_r$ generators of J s.t. $R \to S$ splits outside $V(f_i)$ for each i.

completely check J-almost-isom by checking f_i-almost-isom for each i.

can assume $S =$ literally split.

- reduce to Galois case by
  - first reducing to constant rank (by stratification)
  - use Galois closure construction.

A stratification lemma

<u>Lemma</u> $R \to S$ module finite, fin pres morphism in ~~Rings~~ Rngs.

$\exists\, g_1, \ldots, g_n \in R$ s.t.

$R_i = R/(g_1, \ldots, g_{i-1})_{red}[g_i^{-1}]$ $\quad S_i = S \otimes_R R_i$

- $\bigsqcup \operatorname{Spec} R_i = \operatorname{Spec} R$ ($\Leftarrow g_1, \ldots, g_n$ generate unit ideal)
- $R_i \to S_i$ factors as $R_i \xrightarrow[\text{étale}]{\text{finite}} T_i \xrightarrow[\text{homeomorphism}]{\text{universal}} S_i$.
- in each R_i, either $\cancel{p = 0}$

  ✓ $p \in R_i^{\times}$.

<u>Pf</u> Reduce to noetherian case.

Prismatic coperfection of an integral extension of a lens

Thm (A, I) = perfect prism with slice R

$R \to S$ derived p-completion of an integral map

Then $\Delta_{S/A,\mathrm{perf}}$ is concentrated in degree 0, where it is a derived p-complete perfect prism over A.

$\Rightarrow$ S_{lens} is a lens concentrated in degree 0.

Pf Induct on #n in previous lemma.

$n = 1$: $R \to S_{\mathrm{red}}$ is finite étale, so S_{red} is a lens.

(Check: $S_{\mathrm{lens}} \to S_{\mathrm{red}}$, lens is an isom (by arc_p-descent))

$n > 1$: arc-descent ...