q-de Rham cohomology

Schedule adjustment: no office hours on Thursday, May 27. To make up for it, I'll have an extra office hour on Thursday, June 10. (Lectures and after-class office hours end Friday, June 4.)

Also, no lecture or office hours on Monday, May 31 (university holiday).

Coperfection of an integral extension of a lens

Let (A, I) = perfect prism, slice = R

$R \to S$ derived p-completion of an integral map

Then $\Delta_{S/A, \text{perf}}$ is concentrated in degree 0, where it is derived p-complete, perfect δ-rings.

$\Rightarrow S_{\text{lens}}$ is concentrated in degree 0, where it's a lens.

Reminder: we know concentration in degree ≥ 0.

Idea: use inductive argument to reduce to: $\exists g \in R$

s.t. $R(g^{-1}) \to S_{\text{red}}(g^{-1})$ is finite étale.

replace S with integral closure of R in $S_{\text{red}}(g^{-1}) = S'$ (apply almost purity to this.)

[Need: negative char S - killed by inverting g and (g)-almost zero,

Almost purity (full version)

R = perfd, J = fin. gen. ideal of R

S = module-finite, fin. presented R-algebra,

$\mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is finite étale away from $V(J)$.

1. $\cdot$ S_{perfd} concentrated in degree 0, where it's a perfd.

$\cdot$ $S \to S_{perfd}$ isom away from $V(J)$

$\cdot$ For $n > 0$, $R/p^n \to S_{perfd}/p^n$ is almost ~~finite~~ étale w.r.t. $(R/p^n, J_{perfd,n})$

subsumes Faltings, Scholze, K-Liu, André perfectoid Abhyankar lemma.

Application: cohomological dimension

Lemma: R = p-torsion-free tors. $R \to$ is derived
p-completion of a finite morphism
mod p finite

Then $\mathrm{Cone}(S_{\mathrm{crys}} \to (\mathcal{S}, \mathcal{J})_{\mathrm{crys}})$ is concentrated in
degrees ≤ 0 (actually $[-1, 0]$)
(both discrete, map is surjective..

$\vdots$ apply étale comparison

Thm $X = \mathrm{Spec}\, R(p^{-1})$. Then X has étale $\mathbb{F}_p$-cohom dim ≤ 1
(improved Artin-Schreier)
(note: related result of Achinger)

Application: the direct summand conjecture

$R = \mathbb{Z}_p[\![x_1, \ldots, x_r]\!]$, $R \to S$ injective, module finite ring hom. Then

this map splits in Mod_R.

Sketch: enough to split mod p^n $\forall n$,

can replace R with faithfully flat extension

$R \to \mathbb{Z}_p(\zeta_{p^\infty})[\![x_1^{p^{-\infty}}, \ldots, x_r^{p^{-\infty}}]\!]$ with p-almost
$= R_1$

$\to R_2$ AIC (André flatness).

Pick $f \in R \neq 0$, $R(f^{-1}) \to S(f^{-1})$ finite étale. $f^{p^{-\infty}} \in R_2$.

apply almost purity $J = (p f)$, unwind,

extension class is killed by $(pf)^{p^{-n}}$.

A brief history of q

$\alpha_1 \dots \alpha_m, \beta_1 \dots \beta_n$ (classical: $m=2$, $n=1$)

hypergeometric series

$${}_mF_n(\underline{\alpha}; \underline{\beta}; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_m)_k}{(\beta_1)_k \cdots (\beta_n)_k} \frac{z^k}{k!}$$

(Pochhammer symbol)

$$(x)_k = x(x+1)\cdots(x+k-1)$$

q-Pochhammer symbol

$$(x;q)_k = \prod_{i=0}^{k-1} (1 - xq^i)$$

q-analogue (Heine)

$\frac{1-xq^i}{1-q} \to x+i$ as $q \to 1$

$${}_m\phi_n(\alpha_1 \dots \alpha_m; \beta_1 \dots \beta_n; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1;q)_k \cdots (\alpha_m;q)_k}{(\beta_1;q)_k \cdots (\beta_n;q)_k (q;q)_k} \left((-1)^k q^{k(k-1)/2} \right)^{1+n-m} z^k.$$

(idea: as $q \to 1$, this resembles previous one)

The q-derivative of Jackson

$$D_q f(x) = \frac{f(qx) - f(x)}{qx - x}$$

(as in power series in $q - 1$, constant coeff is usual $f'(x)$)

$$D_q(f+g) = D_q f + D_q g$$

$$D_q(f(x)g(x)) = \frac{f(x)g(qx) - f(x)g(qx) + f(x)g(qx) - f(x)g(x)}{\ldots}$$

$$f(x)(D_q g)(x) + (D_q f)(x)\, g(qx).$$

(or vice versa)

The q-integral of Jackson

$$\int f(x)\,d_q x = (1-q)x\sum_{n=0}^{\infty} q^k f(q^k x)$$

in some context where this makes sense.

Context: the q-hypergeometric differential equation

want to view q-hypergeom series
as satisfying q-analogue of diff eq
for hypergeom series

$$D = z\frac{d}{dz} \quad \Big(z \prod_i (D+\alpha_i) - \prod_j (D+\beta_j - 1) \Big) (\ \ ) = 0.$$

Picard-Fuchs equation of interest in many cases.

(I'm interested in p-adic cohomology here..)

The framed q-de Rham complex of a polynomial ring

$$q\Omega^\bullet_{R[x]/R,\square} = \left(R[x][[q-1]] \xrightarrow{\nabla_q} R[x][[q-1]]\,dx \right)$$

$$* \mapsto D_q(*)\cdot dx$$

$$q\Omega^\bullet_{R[x_1,\dots,x_r]/R,\square} = q\Omega^\bullet_{R[x_1]/R,\square} \boxtimes \cdots \boxtimes q\Omega^\bullet_{R[x_r]/R,\square}$$

mod $q-1$, get usual dR complex.

The situation over Q

If $\mathbb{Q} \subseteq R$, then this is is an iso usual dR complex tensored up to $R[[q-1]]$

In particular, get to eliminate coordinate dependence.

The dga structure on the q-de Rham complex

get a DGA structure

by giving $q\Omega^1$

the bimodule structure over $q\Omega^0$

using usual action on LHA

and q-twisted action on other side.

The failure of functoriality

One one hand this construction of $q\Omega^\bullet$ is not coordinate independent (e.g. $x \mapsto x+1$)

However, we will prove that underlying object in $D(\mathbb{R}[[q-1]])$ is a well-defined functorial commutative algebra object.

(Scholze, Bhatt-Scholze, Pridham)