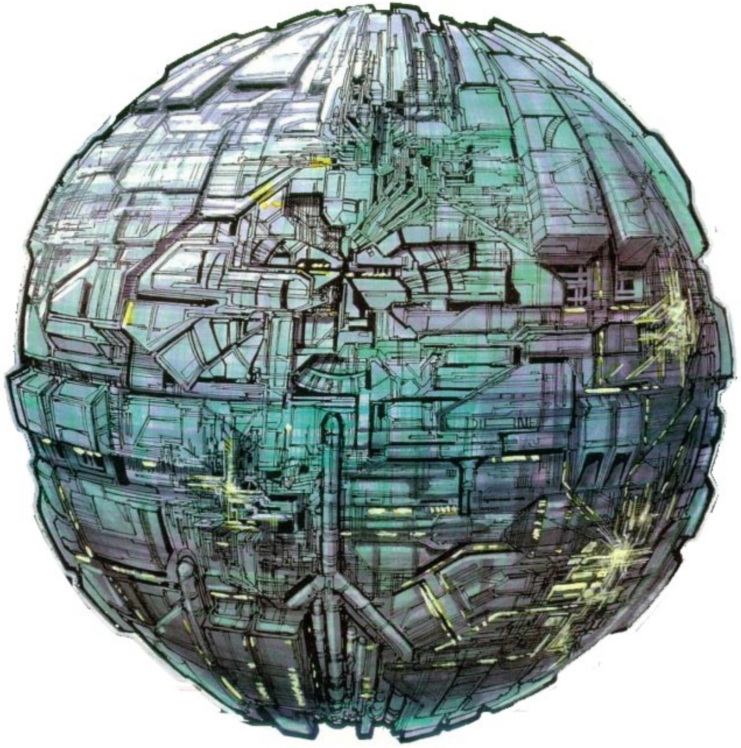
q-crystalline cohomology

Schedule adjustment: no office hours on Thursday, May 27. To make up for it, I'll have an extra office hour on Thursday, June 10. (Lectures and after-class office hours end Friday, June 4.)

Also, no lecture or office hours on Monday, May 31 (university holiday).



A δ-ring structure

$A = \mathbb{Z}_p[[q-1]]$ is a δ-ring with $q = (q-1)+1$ constant.

$(A, (p)_q)$ is a prism (but $(A, q-1)$ is not a prism)

$$(p)_q = \frac{1-q^p}{1-q}$$

(note: $(p, q-1)$-topology = $(p, (p)_q)$-topology)

$A/(p)_q = \mathbb{Z}_p(\zeta_p)$ via $q \mapsto \zeta_p$.

(give later an analogue of "Frobenius twist")

How not to take *q*-divided powers

$$[n]_q! = [1]_q \cdots [n]_q$$

$$\gamma_{n,q}(x) \stackrel{?}{=} \frac{x^n}{[n]_q!}$$

$$\gamma_{n,q}(x+y) = \sum_{i=0}^{n} \underbrace{\frac{n!\,[i]_q!\,[n-i]_q!}{[n]_q!\; i!\,(n-i)!}}_{\equiv 1 \bmod q.} \gamma_{i,q}(x)\,\gamma_{n-i,q}(y)$$

basically unusable definition.

A proxy for the q-divided p-th power

$D = (p)_q$-torsion-free δ-rings over A

$$x^p = \phi(x) - p\delta(x)$$

Define

$$\gamma(x) = \frac{\phi(x)}{(p)_q} - \delta(x)$$

for $x \in D$ with $\phi(x) \in (p)_q D$.

Note: $\gamma(x+y) = \gamma(x) + \gamma(y) - \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} x^i y^{p-i}$.

$\gamma(xy) = \phi(y)\gamma(x) - x^p \delta(y)$.

A criterion for q-divided powers

$I \subset D$ ideal

$\Rightarrow J = \{x \in I : \phi(x) \in (p]_q D,\ \gamma(x) \in I\}$

is an ideal of D.

say D has <u>q-divided powers</u> on I if $I = J$.

<u>Lemma</u> the ideal $\phi^{-1}((p]_q D)$ admits q-divided powers

idea of proof: similar calculation as in classical case, where you first reduce to "universal" setting.

$A\{x, \phi(x)/(p)_q\}^{\wedge}_{(p,(p)_q)}$

A flatness lemma

$D = A\langle x, \frac{\phi(x)}{[p]_q}\rangle^{\wedge}_{(p,[p]_q)}$

$\phi([p]_q) \equiv p \mod [p]_q$

mod $D/[p]_q$ is p-torsion-free.

$\phi(\gamma(x)) \in [p]_q D \Leftarrow \phi([p]_q)\,\phi(\gamma(x)) \in [p]_q D.$

calculate ...

this

<u>lemma</u>: $A \to A\langle x_1 \ldots x_r, \frac{\phi(x_1)}{[p]_q}, \ldots, \frac{\phi(x_r)}{[p]_q}\rangle^{\wedge}_{(p,[p]_q)}$

is $(p,[p]_q)$-completely flat.

<u>Pf</u> Calculation, after $\mathbb{Z}_p[[q-1]] \to \mathbb{Z}_p[[q^{1/p}-1]]$

q-pd pairs

(D, I) D = δ-ring over $A = \mathbb{Z}_p[[q-1]]$
I = ideal

- D, D/I derived $(p, [p]_q)$-complete
- $q-1 \in I$, $\phi(I) \subseteq [p]_q D$, $\gamma(I) \subseteq I$
  "D admits q-divided powers on I"
- D is $[p]_q$-torsion-free, $D/[p]_q$ bounded p-power torsion
  $\Leftrightarrow$ $(D, [p]_q)$ is a bounded prism over $(A, [p]_q)$
- $D/(q-1)$ is p-torsion-free, finite $(p, [p]_q)$-Tor amplitude over D.
  (e.g. D is $(p, [p]_q)$-completely flat over A)

q-divided power envelopes

(B = perfect δ-ring over A derived $(p,[p]_q)$-complete $(B, \phi(([p]_q))$'s q-pd-pair.

$P \sim P$

$P = \delta$-ring over A

$P \twoheadrightarrow R = \mathbb{Z}_p\langle X_1, \dots, X_r\rangle^{\wedge}_{(p)}$ kernel $\bar{J} = (q-1, y_1, y_2, \dots)$

$y_1, y_2, \dots$ regular sequence $P/(p, q-1)$.

$$D = P\left\{ \frac{\phi(y_i)}{[p]_q} \dots \right\}^{\wedge}_{(p,[p]_q)}$$

← q-pd envelope of (P, J)

$I := \ker(D \to D/(q-1) \to R)$

Claim: (D, I) is a q-pd-pair with a map $(P, J) \to (D, I)$ of δ-pairs and is universal for this property.

Note: mod $q-1$, recovers usual pd envelope.

The Hodge-Tate comparison "magic diagram"

$P \to R \quad x_i \mapsto x_i \quad \delta^m(x_i) \to 0 \ (m \to \infty).$

$P = A\{x_1, \dots, x_r\}^{\wedge}_{(p,(r)_q)}$

$R = \mathbb{Z}_p[x_1, \dots, x_r]^{\wedge}_{(p)}$

Čech-Alexander

$$
\begin{tikzcd}
D_{J^0,q}(P^0) \arrow[r] \arrow[d] & D_{J^1,q}(P^1) \arrow[r] \arrow[d] & D_{J^2,q}(P^2) \arrow[r] \arrow[d] & \cdots \\
D_{J^0,q}(P^0)\widehat{\otimes}_{P^0} q\widehat{\Omega}^1_{P^0/\mathbb{Z}_p,\square} \arrow[r] \arrow[d] & D_{J^1,q}(P^1)\widehat{\otimes}_{P^1} q\widehat{\Omega}^1_{P^1/\mathbb{Z}_p,\square} \arrow[r] \arrow[d] & D_{J^2,q}(P^2)\widehat{\otimes}_{P^2} q\widehat{\Omega}^1_{P^2/\mathbb{Z}_p,\square} \arrow[r] \arrow[d] & \cdots \\
D_{J^0,q}(P^0)\widehat{\otimes}_{P^0} q\widehat{\Omega}^2_{P^0/\mathbb{Z}_p,\square} \arrow[r] \arrow[d] & D_{J^1,q}(P^1)\widehat{\otimes}_{P^1} q\widehat{\Omega}^2_{P^1/\mathbb{Z}_p,\square} \arrow[r] \arrow[d] & D_{J^2,q}(P^2)\widehat{\otimes}_{P^2} q\widehat{\Omega}^2_{P^2/\mathbb{Z}_p,\square} \arrow[r] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}
$$

except for row 0 rows acyclic for combinatorial reasons.

The q-Poincaré lemma via derived Nakayama

Each column is quasi-isom to $\widehat{\Omega}^{\bullet}_{R/\mathbb{Z}_p,\square}$ (and in particular to each other).

Interpretation of the top row via the q-crystalline site

Top row computes cohom of q-crystalline site

(opposite category of)

q-pd pairs (D, I) over $(A, (\phi_q))$ + isomorphism

$$D/I \cong R.$$

$\Rightarrow$ coordinate-free interpretation of

$$q\widehat{\Omega}^{\bullet}_{R/\mathbb{Z}_p,\square} \text{ in } D(A)$$

A "Frobenius twist"

$R = \mathbb{Z}_p(x, \quad - x_n)^\wedge_{(p)}$

$$R^{(1)} = R \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\zeta_p]$$

(D, I) q-pair with $D/I \cong R$

$$\phi(I) \subseteq (p)_q D$$

$$\Rightarrow R = D/I \to D/(p)_q$$

linear over Frobenius on A

$$A \otimes_{\phi, A} R \to D/(p)_q$$

q-crys site

$$\searrow (A \otimes_{\phi, A} R)/(p)_q \cong R^{(1)} \nearrow$$

$\Rightarrow$ prismatic site of $R^{(1)}$

Prismatic cohomology and q-de Rham cohomology

Get $\Delta_{R^{(1)}/A} \cong \{\Omega^{\bullet}_{R/\mathbb{Z}_p}, D\}$

canonical in $D(A)$

"Cartier isom"

q-H-Tate:

$$H^{\bullet}(q\Omega_{R/\mathbb{Z}_p} \hat{\otimes}^{L} A/(p)_q) \cong \Omega^{\bullet}_{R/\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[[\zeta_p]]$$

of smooth algs over

$$A/(p)_q \cong \mathbb{Z}_p(\zeta_p)$$

Extension via étale localization and Kan extension

can do étale localization
do Kan extension

these agree for p-completely smooth $\mathbb{Z}_p$-algs.

q-crys cohom mod $q-1$
(for X smooth proper recovers usual crystalline cohom.
(formal scheme/$\mathbb{Z}_p$).

Frobenius is an isogeny

From calculation,

ϕ is an isomorphism away from $(p)_q$