Some further developments: a whirlwind tour

Last lecture Friday, June 4. Office hours also end that day except as noted below.

The office hours on Thursday, June 3 will start 30 minutes later than usual (10:30am PDT).

I will have an additional office hour on Thursday, June 10 at the usual time (10:00am PDT).

Hochschild homology

$A \to B$ morphism of $\underline{\text{Rings}}$

$K_\bullet$ = simplicial object in $\underline{\text{Rings}}_A$

$$K_n = \underbrace{B \otimes_A B \otimes_A \cdots \otimes_A B}_{n+1}$$

$$K_{n+1} \to K_n \qquad B \otimes \cdots \underbrace{B \otimes B}_{\downarrow m} \cdots \otimes B$$

$$B \otimes \cdots \quad B \otimes \cdots B$$

associated complex (homological numbering)

$\Rightarrow$ Hochschild homology.

Topological Hochschild homology (THH)

Rings ⟶ category of E_∞-rings spectra

initial object sphere spectrum

contains Top + (suspension)

{sphere spectrum} ⟶ $\mathbb{S}$

related to p-adic Hodge theory (Hesselholt-Madsen)

Will appears in Bhatt-Morrow-Scholze 2

and is further promoted by prismatic setup.

The absolute prismatic site

$(\mathrm{Spec}\, R)^{op}_{\Delta}$

R = derived p-complete

absolute prismatic (o)(site of R

= category of prisms (B, J) equipped with

ring hom $R \to B/J$ of

(no base ~~prism~~!)

morphisms are morphisms $(B, J) \to (B', J')$

s.t. $B/J \longrightarrow B'/J'$ ring

is R-linear.

$$\mathcal{O} : (B, J) \longrightarrow B$$

$$\overline{\mathcal{O}} : (B, J) \longrightarrow B/J$$

Prismatic *F*-crystals

A crystal on this site consists of:

for each object X, finite projective $\mathcal{O}(X)$-module

+ for each morphism $Y \to X$ a rigidity property

$$M(X) \otimes_{\mathcal{O}(X)} \mathcal{O}(Y) \xrightarrow{\sim} M(Y).$$

F-crystal: a crystal M + isom

$$\varphi^* M(I^{-1}) \longrightarrow M(I^{-1})$$

Prismatic *F*-crystals and crystalline lattices

Thm (Bhatt-Scholze)

K = complete discretely valued field of mixed char $(0, p)$
perfect residue field

Then category of prismatic F-crystals on $\mathcal{O}_K$

$\simeq$ category of crystalline lattices $\mathbb{Z}_p(1)$

i.e. finite free $\mathbb{Z}_p$-modules
+ continuous G_K-action

s.t. base extension from $\mathbb{Z}_p$ to $\mathbb{Q}_p$
is a crystalline Galois representation
sense of Fontaine.

The Cartier-Witt stack of Bhatt-Lurie

A = derived p-complete rings

prism structures (A, I) $\longleftrightarrow$ morphisms (prismatization) $\operatorname{Spec} A \to CW$

Cartier-Witt "stack"

$W = \operatorname{Spec} \mathbb{Z}(x_0, x_1, \dots)$ represents $A \mapsto W(A)$

$= \operatorname{Spec} \mathbb{Z}[x_0, \dots, x_n, \dots]$

$W_{prim,n}$ = completion of W_n along locally closed subscheme $p = x_0 = 0$, $x_1 \neq 0$

identifies the generic element of $W(\cdot)$ is distinguished.

$CW_n = W_{prim,n} / W_n^{\times}$

$CW = \lim_{n} CW_n$

Quasi-syntomic rings

(want to describe p-divisible groups over such base rings)

$R \in \underline{\mathrm{Rings}}$

R = <u>quasi-syntomic</u> if R = derived p-complete w/ bounded p-power torsion

and $L_{R/\mathbb{Z}_p}$ has p-complete Tor-amplitude in $[-1, 0]$.

(e.g. noetherian + local complete intersection + p-complete

<u>or</u> regular semiperfectoids w/ bounded p-power torsion)

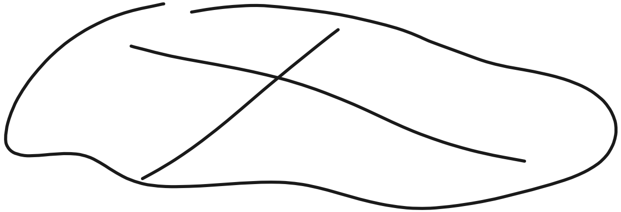
Prismatic Dieudonné theory of Anschutz-Le Bras

Thm For R quasisyntomic,

{p-divisible groups over R} $\xrightarrow{\text{anti}}$ {prismatic Dieudonné crystals over R}

extends both classical (crystalline) Dieudonné theory and Dieudonné theory for perfectoids (e.g. Scholze-Weinstein)

Prelog structures



on a ring A is

Monoid M + morphism $\alpha : M \to A$.

"sheafify" $\to$ log structure.

Koshikawa's δ_log-rings

$$(\underbrace{A, \delta}_{\text{Rings } \delta}, \underbrace{\alpha : M \to A}_{\text{prelog structure}}, \delta_{\log})$$

logarithmic p-derivative

$$\delta_{\log} : M \longrightarrow A$$

$$\delta_{\log}(e) = 0$$

$$\delta(\alpha(m)) = \alpha(m)^p \delta_{\log}(m)$$

$$\delta_{\log}(m m') = \delta_{\log}(m) + \delta_{\log}(m') + p\, \delta_{\log}(m)\, \delta_{\log}(m')$$

The point here: semistable Breuil-Kisin modules

Usual prismatic cohomology gives you

a cohomology theory on smooth proper

$\mathcal{O}_K$-schemes

$(K : \mathbb{Q}_p) < \infty$

valued in Breuil-Kisin modules

log discussion can be used similarly for

smooth proper K-schemes

with semistable reduction

(Hyodo-Kato theory)

Title of this slide

Comment: another approach is <u>stacky</u>:

log structures can be described using $\underline{A^1/\mathbb{G}_m}$