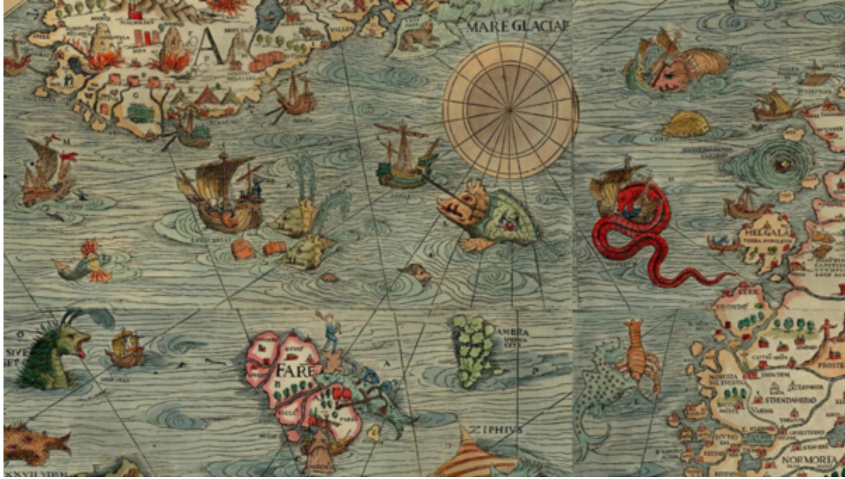
Some global speculation



Last lecture! Thanks for participating, and especially for your feedback on the lecture notes.

Reminder: extra office hour Thursday, June 10 starting at 10am PDT (UTC-7).

Zulip will remain available for questions and discussion. I will keep monitoring it through the summer, and will update the lecture notes with corrections and clarifications (but probably not any new material).

UCSD students: don't forget to submit a course evaluation by Monday, June 7. Thank you!

Divided power envelopes of δ-rings revisited

$R_0 \in \underline{\text{Ring}}$ $\mathbb{Z}$-flat

R = free δ-ring on x over R_0 (used to write $R_0\{x\}$)

J = ideal of R generated by $(x, \delta(x), \delta^2(x), \dots)$

Then map from R to pd-envelope of (R, J) promotes to a morphism of δ-rings.

$\underline{\text{Pf}}$ D = pd-envelope $\subseteq R \otimes_{\mathbb{Z}} \mathbb{Q}$ has a largest ideal on which standard divided powers

contains $\delta^m(x)$, and hence: $\delta^m(x)^p + p\,\delta^{m+1}(x) = \phi(\delta^m(x))$

so $\phi : D \to D$. This induces a Frobenius lift ... i.e.

$m \geq 0$, $n \geq 1$ $\quad \phi(\gamma_n(\delta^m(x))) \equiv \gamma_n(\delta^m(x))^p \equiv 0 \pmod{p}$

Reminder: λ-rings and their Frobenius lifts

$R \in \underline{\text{Ring}}$ $\quad \mathbb{Z}$-flat

A $\underline{\lambda\text{-ring structure}}$ on R is equivalent to a commuting family of p-Frobenius lifts ψ^p for each prime p.

get maps $\lambda_n : R \to R$

example for later: $\mathbb{Z}[[q-1]]$ $\qquad \psi^p(q) = q^p$

Divided power envelopes of λ-rings

$R_0 \in \underline{\mathrm{Ring}}$ $\mathbb{Z}$-flat

$R_0(x, \lambda_2(x), \lambda_3(x), \dots)$, $x = \lambda_1(x)$

$R = R_0\{x\}$ ← free λ-ring on R_0

(i.e. $R_0[\lambda_m(x) : m \geq 1]$)

J = ideal of R generated by $(\lambda_m(x))_{m \geq 1}$

Then map from R to pd-envelope of (R, J) promotes to a morphism of λ-rings.

<u>Pf</u> As above, D = pd envelope; for each p, $\psi^p : R \to R$ extends to D, and is a Frobenius lift (and they still commute)

A corollary

In Nomura's setup, pd-envelope has form

$$D = R\left\{ \frac{\psi^p(x)}{p} : p \text{ prime} \right\}$$

$= D'$

(pf: check $\gamma_n(\delta^m(x)) \in D'$ by induction on n.)

Another look at the magic diagram

$R_0 \in \mathrm{Ring}$ $\quad R = R_0[x_1, \dots x_r]$ viewed as ∂-ring with $x_1 \cdots x_r$ constant.

$P^n = R_0\{x_{ij} : i = 1 \dots r, \ j = 0 \dots n\}$ $\leftarrow$ ∂-adjunction.

$J^n = \ker(P^n \to R)$ $\quad x_{ij} \mapsto x_i \quad \partial^m(x_r) \to 0 \quad (m \geq 1)$

columns are all quasi-iso by Poincaré lemma

$$\begin{tikzcd}
 & D_{J^0}(P^0) \arrow[r] \arrow[d] & D_{J^1}(P^1) \arrow[r] \arrow[d] & D_{J^2}(P^2) \arrow[r] \arrow[d] & \cdots \\
\Omega^\bullet_{R/R_0} \arrow[r] & D_{J^0}(P^0) \otimes_{P^0} \Omega^1_{P^0/R_0} \arrow[r] \arrow[d] & D_{J^1}(P^1) \otimes_{P^1} \Omega^1_{P^1/R_0} \arrow[r] \arrow[d] & D_{J^2}(P^2) \otimes_{P^2} \Omega^1_{P^2/R_0} \arrow[r] \arrow[d] & \cdots \\
 & D_{J^0}(P^0) \otimes_{P^0} \Omega^2_{P^0/R_0} \arrow[r] \arrow[d] & D_{J^1}(P^1) \otimes_{P^1} \Omega^2_{P^1/R_0} \arrow[r] \arrow[d] & D_{J^2}(P^2) \otimes_{P^2} \Omega^2_{P^2/R_0} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

rows acyclic

A λ-ring structure on Z[[q-1]]

$A = \mathbb{Z}[[q-1]]$

λ-ring with q constant (i.e. $\psi^n(q) = q^n$ $\forall n$)

A q-divided p-th power construction

D = A-torsion-free λ-ring over A

p = prime,

$x \in D$ $\quad \psi^p(x) \in \underbrace{(p)_q}_{q^{p-1}+\dots+1 \;=\; (q^p-1)/(q-1)} D$,

write $\gamma_{p,q}(x) = \dfrac{\psi^p(x)}{(p)_q} - \delta_p(x) \in D$.

<u>Lemma</u>

ideal $(\psi^p)^{-1}((p)_q D)$ is stable under $\gamma_{p,q}$.

(as before).

λ-pairs and global q-pd pairs

λ-pair = pair (D, I) $\quad$ D = λ-rings over A
I = ideal of D.

morphism $(D, J) \to (E, J)$ is a morphism of λ-rings $D \to E$ carrying I into J.

global q-pd pair = λ-pair (D, I)
D = derived $(q-1)$-complete, $I \ni q-1$
and for each p, $\psi^p(I) \subseteq (p)_q D$
and $\gamma_{p,q}(I) \subseteq I$.

(note: $(A, q-1)$ is initial object.)

Global q-pd envelopes

pnp $P = \lambda$-ring over A with surjection of rings $P \to R = \mathbb{Z}(x_1, \dots x_r)$

kernel $J = (q-1, y_1, y_2, \dots)$ where $y_1, y_2 \dots$ is a regular sequence in $P/(q-1)$

Define λ-ring

$$D = P\left\{ \frac{\psi_p(y_i)}{(p)_q} : \begin{array}{l} \text{all } i \\ \text{all primes } p \end{array} \right\}^{\wedge}_{(q-1)}$$

$I = \ker(D \to D/(q-1) \to R)$

Then $P/J \cong D/I$; $D/q-1 = $ pd-envelope of $(P/(q-1), \operatorname{im}(J))$;

and $(P, J) \to (D, I)$ of λ-pairs is universal for morphism target a global pd-pair.

One last look at the magic diagram

$R_0 = \mathbb{Z}$ $\quad R = \mathbb{Z}(x_1, \dots, x_r)[[q-1]]$ ← δ-rings with $q, x_1 \dots x_r$ all constant

$P^n = \mathbb{Z}\{x_{i,j} : i = 1 \dots r, j = 0 \dots n\}[[q-1]]$

$J^n = \ker(P^n \to R)$ $\quad x_{i,j} \mapsto x_i$, $\delta^m(x_{i,j}) \mapsto 0 \ \forall m > 0$.

$D_{J^n,q}(P^n) =$ global q-pd-envelope of (P^n, J^n).

$q\widehat{\Omega}^\bullet_{R/\mathbb{Z},\square} \to$

$$\begin{array}{ccccccc}
D_{J^0,q}(P^0) & \longrightarrow & D_{J^1,q}(P^1) & \longrightarrow & D_{J^2,q}(P^2) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
D_{J^0,q}(P^0)\widehat{\otimes}_{P^0} q\widehat{\Omega}^1_{P^0/\mathbb{Z},\square} & \longrightarrow & D_{J^1,q}(P^1)\widehat{\otimes}_{P^1} q\widehat{\Omega}^1_{P^1/\mathbb{Z},\square} & \longrightarrow & D_{J^2,q}(P^2)\widehat{\otimes}_{P^2} q\widehat{\Omega}^1_{P^2/R_0,\square} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
D_{J^0,q}(P^0)\widehat{\otimes}_{P^0} q\widehat{\Omega}^2_{P^0/\mathbb{Z},\square} & \longrightarrow & D_{J^1,q}(P^1)\widehat{\otimes}_{P^1} q\widehat{\Omega}^2_{P^1/\mathbb{Z},\square} & \longrightarrow & D_{J^2,q}(P^2)\widehat{\otimes}_{P^2} q\widehat{\Omega}^2_{P^2/R_0,\square} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

The global *q*-crystalline site

$R = \mathbb{Z}[x_1, \ldots, x_r]$

global q-crystalline (big) site to be

category global q-pd-pairs (P, J)

+ isom of rings $P/J \cong R$ (no λ-structure).

as in p-local case

topology of previous diagram

is a Čech-Alexander complex for this.

Okay, now what?

Conclusion: $q\text{-}\widehat{\Omega}_{\mathbb{Z}[x_1,\ldots,x_n]/\mathbb{Z},\square}$ is canonically independent of coordinates as object of $D(\mathbb{Z}[[q-1]])$.
(recovers a result of Pridham)

What can one do with this???

Title of this slide

Warnings: natural next step: compare etale localization with left Kan extension

subtlety: our derived interpretation of de Rham (cohomology used existence of conjugate filtration in char p. The analogue in char 0 is more subtle (affects Hodge-Tate comparison)

Title of this slide

Is there an analogue of the notion of a <u>prism</u> in λ-ring setup?

e.g. $\left(\mathbb{Z}_p[[q-1]], ([p]_q)\right)$

$$\rightsquigarrow$$

$\left(\mathbb{Z}[[q-1]], (q-1), ([2]_q), ([3]_q), \dots\right.$