ÉTALE AND CRYSTALLINE COMPANIONS, II

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ABSTRACT. Let $X$ be a smooth scheme over a finite field of characteristic $p$. In answer to a conjecture of Deligne, we establish that for any prime $\ell \neq p$, an $\ell$-adic Weil sheaf on $X$ which is algebraic (or irreducible with finite determinant) admits a crystalline companion in the category of overconvergent $F$-isocrystals, for which the Frobenius characteristic polynomials agree at all closed points (with respect to some fixed identification of the algebraic closures of $\mathbb{Q}$ within fixed algebraic closures of $\mathbb{Q}_\ell$ and $\mathbb{Q}_p$). The argument depends heavily on the free passage between $\ell$-adic and $p$-adic coefficients for curves provided by the Langlands correspondence for $\mathrm{GL}_n$ over global function fields (work of L. Lafforgue and T. Abe), and on the construction of Drinfeld (plus adaptations by Abe–Esnault and the author) giving rise to étale companions of overconvergent $F$-isocrystals.

INTRODUCTION

0.1. Overview. Throughout this paper, let $k$ be a finite field of characteristic $p$ and let $X$ be a smooth scheme over $k$. In a previous paper [37], we studied the relationship between coefficient objects (of locally constant rank) in Weil cohomology with $\ell$-adic coefficients for various primes $\ell$. For $\ell \neq p$, such objects are lisse Weil $\mathbb{Q}_\ell$-sheaves, while for $\ell = p$ they are overconvergent $F$-isocrystals.

The purpose of this paper is to complete the proof of a conjecture of Deligne [15, Conjecture 1.2.10] which asserts that all coefficient objects “look motivic”, that is, they have various features that would hold if they were to arise in the cohomology of some family of smooth proper varieties over $X$. The deepest aspect of this conjecture is the fact that coefficient objects do not occur in isolation; in a certain sense, coefficient objects in one category have “companions” in the other categories.

To make this more precise, we say that an $\ell$-adic coefficient object $\mathcal{E}$ on $X$ is algebraic if for each closed point $x \in X$, the characteristic polynomial of (geometric) Frobenius acting on the fiber $\mathcal{E}_x$ has coefficients which are algebraic over $\mathbb{Q}$. (These coefficients are then contained in a single number field independent of $x$; see [37, Theorem 1.1].)

To define the companion relation, consider two primes $\ell, \ell'$ and fix an identification of the algebraic closures of $\mathbb{Q}$ within $\mathbb{Q}_\ell$ and $\mathbb{Q}_{\ell'}$. We say that an algebraic $\ell$-adic coefficient object $\mathcal{E}$ on $X$ and an algebraic $\ell'$-adic coefficient object $\mathcal{E}'$ on $X$ are companions if for each closed...
point \( x \in X \), the characteristic polynomials of Frobenius on \( E_x, E'_x \) coincide; note that \( E \) then determines \( E' \) up to semisimplification [37, Theorem 5.2].

With this definition, we can state a theorem answering [15, Conjecture 1.2.10] (and more precisely [37, Conjecture 0.1.1] or [10, Conjecture 1.1]), incorporating Crew’s proposal [12, Conjecture 4.13] to interpret (vi) by reading the phrase “petit camarade cristalline” to mean “companion in the category of overconvergent \( F \)-isocrystals.” (The definition of an overconvergent \( F \)-isocrystal was unavailable at the time of [15]; it was subsequently introduced by Berthelot [6].) See Corollary 8.1.4 for the proof.

**Theorem 0.1.1.** Let \( E \) be an \( \ell \)-adic coefficient object which is irreducible with determinant of finite order. (Recall that we allow \( \ell = p \) here.) In the following statements, \( x \) is always quantified over all closed points of \( X \), and \( \kappa(x) \) denotes the residue field of \( x \).

(i) \( E \) is pure of weight 0: for every algebraic embedding of \( \overline{\mathbb{Q}}_\ell \) into \( \mathbb{C} \) and all \( x \), the images of the eigenvalues of \( F_x \) all have complex absolute value 1.

(ii) For some number field \( E \), \( E \) is \( E \)-algebraic: for all \( x \), the characteristic polynomial of \( F_x \) has coefficients in \( E \). (Beware that the roots of this polynomial need not belong to a single number field as \( x \) varies.)

(iii) \( E \) is \( p \)-plain: for all \( x \), the eigenvalues of \( F_x \) have trivial \( \lambda \)-adic valuation at all finite places \( \lambda \) of \( E \) not lying above \( p \).

(iv) For every place \( \lambda \) of \( E \) above \( p \) and all \( x \), every eigenvalue of \( F_x \) has \( \lambda \)-adic valuation at most \( \frac{1}{2} \text{rank}(E) \) times the valuation of \#\( \kappa(x) \).

(v) For any prime \( \ell' \neq p \), there exists an \( \ell' \)-adic coefficient object \( E' \) which is irreducible with determinant of finite order and is a companion of \( E \).

(vi) As in (v), but with \( \ell' = p \).

Of the various aspects of Theorem 0.1.1, all were previously known for \( X \) of dimension 1, and all but (vi) for general \( X \) (see below); consequently, the new content can also be expressed as follows (answering [37, Conjecture 0.5.1]). See Corollary 8.1.2 for the proof.

**Theorem 0.1.2.** Any algebraic \( \ell \)-adic coefficient object on \( X \) admits a \( p \)-adic companion.

In the remainder of this introduction, we summarize the preceding work in the direction of Theorem 0.1.1, including the results of [37]; we then describe the new ingredients in this paper that lead to a complete proof. See also the survey article [36] for background on \( p \)-adic coefficients.

### 0.2. Prior results: dimension 1

One approach to proving Theorem 0.1.2 would be to show that every \( \ell \)-adic coefficient arises as a realization of some motive, to which one could then apply the \( \ell' \)-adic realization functor to obtain the \( \ell' \)-adic companion. As part of the original formulation of [15, Conjecture 1.2.10], Deligne pointed out that for \( X \) of dimension 1, one could hope to execute this strategy for \( \ell, \ell' \neq p \) by establishing the Langlands correspondence for \( \text{GL}_n \) over the function field of \( X \) for all positive integers \( n \); at the time, this was done only for \( n = 1 \) by class field theory, and for \( n = 2 \) by the work of Drinfeld [17]. An extension of Drinfeld’s work to general \( n \) was subsequently achieved by L. Lafforgue [40], which yields parts (i)–(v) of Theorem 0.1.1 when \( \text{dim}(X) = 1 \) except with a slightly weaker inequality in part (iv); this was subsequently improved by V. Lafforgue [41] to obtain (iv) as written (and a bit more).
This work necessarily omitted cases where $\ell = p$ or $\ell' = p$ due to the limited development of $p$-adic Weil cohomology (rigid cohomology) at the time. Building on recent advances in this direction, Abe [1] has replicated Lafforgue’s argument in $p$-adic cohomology; this completes parts (i)–(v) of Theorem 0.1.1 when $\dim(X) = 1$ by adding the cases where $\ell = p$ or $\ell' = p$.

0.3. Prior results: higher dimension. In higher dimensions, no general method for associating motives to coefficient objects seems to be known. The proofs of the various aspects of Theorem 0.1.1 for general $X$ thus proceed by using the case of curves as a black box.

To begin with, suppose that $\ell \neq p$. To make headway, one first shows that irreducibility is preserved by restriction to suitable curves; this was shown by Deligne [16, §1.7] by correcting an argument of L. Lafforgue [40, §VII]. Since parts (i), (iii), (iv) of Theorem 0.1.1 are statements about individual closed points, they follow almost immediately.

As for part (ii) of Theorem 0.1.1, by restricting to curves one sees that the coefficients in question are all algebraic, but one needs a uniformity argument over these curves to show that the extension of $\mathbb{Q}$ generated by all of the coefficients is finite. Such an argument was provided by Deligne [16]. Building on this, Drinfeld [18] was then able to establish part (v) of Theorem 0.1.1 using an idea of Wiesend [66] to patch together tame Galois representations based on their restrictions to curves. (Esnault–Kerz [21] refer to this technique as the method of skeleton sheaves.)

It is not entirely automatic to extend these results to the case $\ell = p$, as several key arguments (notably preservation of irreducibility) are made in terms of properties of $\ell$-adic sheaves with no direct $p$-adic analogues. Generally, arguments that refer to residual representations do not transfer (although there are some crucial exceptions), whereas arguments that refer only to monodromy groups or cohomology do transfer. With some effort, one can replace the offending arguments with alternates that can be ported to the $p$-adic setting, and thus recover the previously mentioned results with $\ell = p$; this was carried out (in slightly different ways) by Abe–Esnault [2] and by this author [37].

Crucially, this possibility of replacement only applies in cases where one starts with a $p$-adic coefficient object; it therefore does not apply to part (vi) of Theorem 0.1.1, where the existence of a $p$-adic coefficient object is itself at issue, and cannot be established using a direct analogue of the $\ell$-adic construction. However, from the above discussion, one can at least deduce some reductions for the problem of constructing crystalline companions; notably, in any given case the existence of a crystalline companion can be checked after pullback along an open immersion with dense image, or an alteration in the sense of de Jong [14]. Also, we may ignore the case $\ell = p$, as we may move from $\ell = p$ to $\ell' = p$ via an intermediate prime different from $p$. This means that in most of this paper, we can focus on the case of an étale $\ell$-adic coefficient object which is tame (that is, whose local monodromy representations are tamely ramified and quasi-unipotent).

0.4. Uniformity, fibrations, and crystalline companions. We now arrive at the methods of the present paper. At a superficial level, the basic strategy is the same as in Drinfeld’s work. Given an étale coefficient object on $X$, the Langlands correspondence implies the existence of a crystalline companion for the restriction to any curve contained in $X$. In order to construct a crystalline companion on $X$ itself, we construct a coherent sequence of mod-$p^n$ truncations whose inverse limit gives rise to the companion. These truncations are constructed by building an analyzing a moduli space parametrizing all possible truncated
coefficient objects, in which one has points arising from the companions on curves; the presence of many such points forces the moduli space to be large enough to give rise to an object over all of $X$.

In order to further this analogy, we need some uniformity properties on companions on curves in order to obtain a useful finiteness property of the moduli spaces of truncations. In both the étale and crystalline cases, one key input is the fact that after passing to a suitable covering of $X$, one can eliminate all wild ramification; in the crystalline case, this requires the semistable reduction theorem for overconvergent $F$-isocrystals [29, 30, 31, 33]. In the étale case, the necessary uniformity assertion is a consequence of properties of tame étale fundamental groups. In the crystalline case, we instead use Harder-Narasimhan polygons together with a recent result of Tsuzuki [62] which allows overconvergent $F$-isocrystals to be reconstructed from the first steps of their (convergent) slope filtrations.

0.5. Applications. We have already mentioned that some of the results of [37] are a posteriori corollaries of Theorem 0.1.2; we recall a few of these at the end of this paper. These include properties of Newton polygons (Theorem 8.2.1, Theorem 8.2.2) and Wan’s theorem on the $p$-adic meromorphy of unit-root $L$-functions (Theorem 8.3.1).

As remarked upon in [37, Remark 2.2.7], the existence of companions on curves suggests a new method for counting lisse Weil sheaves on curves, by directly relating these counts to the zeta functions of moduli spaces of vector bundles (see op. cit. for some references on this question). Theorem 0.1.2 in turn provides an opportunity (albeit one not acted upon here) to make similar arguments on higher-dimensional varieties, where techniques based on the Langlands correspondence (which form the basis for most prior work on the one-dimensional case) do not apply, although one can at least use them to establish a finiteness result [21, Theorem 1.1].

We expect many additional applications to arise in due course, some of which do not explicitly refer to any $p$-adic behavior. For example, the existence of companions should lead to improvements to the work of Krishnamoorthy–Pál on the existence of abelian varieties associated to $\ell$-adic representations [38]. It is an intriguing open question whether the existence of companions can be used to make even further progress on the existence of motives associated to étale or crystalline coefficient objects.

1. Background on algebraic stacks

We now define the moduli stacks of crystals that will play a crucial role in our construction of crystalline companions. Regarding algebraic stacks, we follow the conventions of the Stacks Project [61, Tag 026M]; a more informal overview can be found in [61, Tag 072I].

Before proceeding, we fix some geometric conventions that run throughout the paper.

Notation 1.0.1. Throughout this paper (unless otherwise specified), let $k$ be a finite field of characteristic $p$; let $K$ denote the fraction field of the ring $W(k)$ of $p$-typical Witt vectors with coefficients in $k$; and let $X$ denote a smooth (but not necessarily geometrically irreducible) separated scheme of finite type over $k$. Let $X^\circ$ denote the set of closed points of $X$.

Definition 1.0.2. By a curve over $k$, we will always mean a scheme which is smooth of dimension 1 and geometrically irreducible over $k$, but not necessarily proper over $k$ (this will be specified separately). A curve in $X$ is a locally closed subscheme of $X$ which is a curve over $k$ in the above sense.
**Definition 1.0.3.** A smooth pair over a base scheme $S$ is a pair $(Y, Z)$ in which $Y$ is a smooth $S$-scheme and $Z$ is a relative strict normal crossings divisor on $Y$; we refer to $Z$ as the boundary of the pair. (Note that $Z = \emptyset$ is allowed.) A good compactification of $X$ is a smooth pair $(X, Z)$ over $k$ with $X$ projective (not just proper) over $k$, together with an isomorphism $X \cong X \setminus Z$; we will generally treat the latter as an identification.

1.1. **Algebraic stacks.** We start with very brief definitions, together with copious pointers to [61] which are probably crucial for making any sense of the definitions.

As a critical link between schemes and stacks, we introduce the category of algebraic spaces as per [61, Tag 025X].

**Definition 1.1.1.** Let $\text{Sch}$ denote the category of schemes. For $S \in \text{Sch}$, let $\text{Sch}_S$ denote the category of schemes over $S$, equipped with the fppf topology.

An algebraic space over $S$ is a sheaf $F$ on $\text{Sch}_S$ valued in sets such that the diagonal $F \to F \times F$ is representable, and there exists a surjective étale morphism $h_U \to F$ for some $U \in \text{Sch}$ (writing $h_U$ for the functor represented by $U$). These form a category in which morphisms are natural transformations of functors, containing $\text{Sch}$ as a full subcategory via the Yoneda embedding $U \mapsto h_U$.

With this definition in hand, we can define algebraic stacks.

**Definition 1.1.2.** As per [61, Tag 026N], by an algebraic stack over $S$, we will mean a stack $\mathcal{X}$ in groupoids over $\text{Sch}_S$ for the fppf topology whose diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by algebraic spaces, and for which there exists a surjective smooth morphism $\text{Sch}_U \to \mathcal{X}$ for some scheme $U$. These form a 2-category as per [61, Tag 03YP], which contains $\text{Sch}_S$ as a full subcategory via the operation $U \mapsto \text{Sch}_U$.

A Deligne-Mumford (DM) stack over $S$ is an algebraic stack $\mathcal{X}$ for which the surjective smooth morphism $\text{Sch}_U \to \mathcal{X}$ can be taken to be étale.

**Remark 1.1.3.** Any property of morphisms of schemes which obeys sufficiently strong locality and descent properties admits a natural generalization for algebraic stacks, which obeys corresponding locality and descent properties and moreover specializes back to the original property when applied to the stacks corresponding to ordinary schemes. For example, such generalizations exist for the properties separated [61, Tag 04YV], finite type [61, Tag 06FR], smooth [61, Tag 075T], and proper [61, Tag 0CL4].

**Lemma 1.1.4.** Let $\mathcal{X}$ be an algebraic stack of finite type over a noetherian scheme $S$. Then for any sequence of closed immersions $\cdots \to X_1 \to X_0 = \mathcal{X}$, there exists an index $i$ such that for all $j \geq i$, $X_{j+1} \to X_j$ is an isomorphism.

**Proof.** Fix a surjective smooth morphism $U \to \mathcal{X}$; then the composition $U \to \mathcal{X} \to S$ is a morphism of finite type in $\text{Sch}$. In particular, since $S$ is noetherian, so is $U$; we may thus apply the usual ascending chain condition to the schemes $U_i := \mathcal{X}_i \times_{\mathcal{X}} U$ to conclude. \hfill $\square$

1.2. **Moduli stacks of stable curves.** As a reminder of the practical meaning of some of the previous definitions, we recall the basic properties of moduli stacks of stable curves.

**Definition 1.2.1.** For $g, n \geq 0$, a stable $n$-pointed genus-$g$ curve fibration (or for short a stable curve fibration) consists of a morphism $f : Y \to S$ and $n$ morphisms $s_1, \ldots, s_n : S \to Y$ (all in $\text{Sch}$) satisfying the following conditions.
• The morphism $f$ is flat, of finite presentation, and proper of relative dimension 1.
• Each geometric fiber of $f$ is reduced and connected, has at worst nodal singularities, and has geometric genus $g$.
• The morphisms $s_1, \ldots, s_n$ are sections of $f$ whose images are pairwise disjoint and do not meet the singular locus of any fiber.
• For each geometric fiber of $f$, each irreducible component of genus 0 (resp. 1) contains at least 3 (resp. 1) points which are either singularities of the fiber or points in the image of some $s_i$.

We refer to the complement of the images of $s_1, \ldots, s_n$ in $Y$ and the singular points of all fibers as the smooth unpointed locus of $f$.

Note that for $f$ as above, $f$ is smooth if and only if each geometric fiber of $f$ is smooth. In this case, we say that $f$ is a smooth $n$-pointed genus-$g$ curve fibration (or for short a smooth curve fibration).

**Remark 1.2.2.** It is customary to refer to the fibers of a stable $n$-pointed genus-$g$ curve fibration as stable curves; however, this is not compatible with our running conventions.

**Definition 1.2.3.** Let $\mathcal{M}_{g,n}$ be the stack over $\text{Sch}$ whose fiber over $S \in \text{Sch}$ consists of stable $n$-pointed genus-$g$ curve fibrations over $S$. This is empty unless $2g + n \geq 3$.

Let $M_{g,n}$ be the substack of $\mathcal{M}_{g,n}$ whose fiber over $S \in \text{Sch}$ consists of smooth $n$-pointed genus-$g$ curve fibrations over $S$.

**Remark 1.2.4.** Note that the definition of the full moduli stack of curves in [61, Tag 0DMJ] requires consideration of families of curves in which the total space is an algebraic space rather than a scheme; this does not change anything over the spectrum of an artinian local ring or a noetherian complete local ring [61, Tag 0AE7], but does make a difference over a more general base [61, Tag 0D5D].

However, this discrepancy does not arise for stable curves: if $S$ is a scheme and $f : Y \to S$ is a stable $n$-pointed genus-$g$ fibration in the category of algebraic spaces, then $Y$ is a scheme. That is because the hypotheses ensure that the relative canonical bundle (for the logarithmic structure defined by $s_1, \ldots, s_n$) is ample, so we may realize $Y$ as a closed subscheme of a particular projective bundle over $S$ (compare [61, Tag 0E6F]).

In the language of stacks, the Deligne-Mumford stable reduction theorem takes the following form.

**Proposition 1.2.5.** The stack $\mathcal{M}_{g,n}$ is a smooth proper DM stack over $\mathbb{Z}$. The stack $M_{g,n}$ is a dense open substack, so it is a smooth separated DM stack over $\mathbb{Z}$.

*Proof.* In the case $n = 0$ this is [61, Tag 0E9C]; the general case is similar. \qed

1.3. **Geometric corollaries of stable reduction.** We next recall some geometric corollaries of the stable reduction theorem. Chief among these is de Jong’s theorem on alterations, although the manner in which it is derived from stable reduction will not be relevant here.

**Definition 1.3.1.** An alteration of a scheme $Y$ is a morphism $f : Y' \to Y$ which is proper, surjective, and generically finite étale. This corresponds to a separable alteration in the sense of de Jong [14].
Proposition 1.3.2 (de Jong). For $X$ smooth of finite type over $k$ (as per our running
convention), there exists an alteration $f : X' \to X$ such that $X'$ is smooth (but not necessarily
geoemtrically irreducible over $k$) and admits a good compactification.

Proof. Keeping in mind that $k$ is perfect, see [14, Theorem 4.1]. □

Corollary 1.3.3. Suppose that $S$ is a smooth scheme of finite type over $k$ and $f : X \to S$ is
a smooth $n$-pointed genus-$g$ curve fibration for some $n, g$ with $2g + n \geq 3$. Then there exist
an alteration $S' \to S$, a good compactification $S' \hookrightarrow \overline{S}$, and a stable $n$-pointed genus-$g$ curve
fibration over $\overline{S}$ whose pullback to $S'$ is $X \times_{S} S' \to S'$.

Proof. By Proposition 1.2.5 (see more precisely [61, Tag 0E98]), for any discrete valuation
ring $R$ with fraction field $F$ and any morphism $\text{Spec} F \to M_{g,n}$, there exists a finite separable
extension $F' \to F$ such that, for $R'$ the integral closure of $R$ in $F'$, the composition $\text{Spec} F' \to
\text{Spec} F \to M_{g,n}$ factors uniquely through the inclusion $\text{Spec} F \to \text{Spec} R$.

Turning to the problem at hand, the original fibration corresponds to a morphism $S \to
M_{g,n}$. By applying the previous paragraph as in [14, 4.17], we obtain an alteration $S' \to S$
such that the composition $S' \to S \to M_{g,n} \to \overline{M}_{g,n}$ factors through some compactification $\overline{S}'$ of $S'$. By applying Proposition 1.3.2, we may further ensure that $\overline{S}'$ is a good compactification of $S'$. The induced morphism $\overline{S}' \to \overline{M}_{g,n}$ corresponds to the desired fibration over $\overline{S}'$. □

We will apply the previous result as follows.

Corollary 1.3.4. For $X$ smooth of finite type over $k$ (as per our running
convention), there exist an alteration $f : X' \to X$, an open dense subscheme $U$ of $X'$, and a diagram

$$
\begin{array}{ccc}
U & \longrightarrow & X' \\
\downarrow & & \downarrow \\
S & \longrightarrow & \overline{S}
\end{array}
$$

in which $(\overline{S}, \overline{S} \setminus S)$ is a smooth pair over $k$, $X' \to \overline{S}$ is a stable curve fibration, and $X' \times_{\overline{S}} S \to S$ is a smooth curve fibration with unpointed locus equal to $U$.

Proof. At any stage in the proof, we are free to replace $X$ with either an alteration or an
open dense subscheme (this includes making a field extension on $k$). We may thus assume
at once that $X$ is quasi-projective; we then proceed as in [4, Proposition 3.3].

Choose a very ample line bundle $\mathcal{L}$ on $X$, put $n := \dim(X)$, and choose a point $z \in X^\circ$. Af

After possibly enlarging $k$, if we make a generic choice of $n$ sections $H_1, \ldots, H_n$ of $\mathcal{L}$ containing
$z$, then the intersection $C$ will be zero-dimensional (but nonempty; see Remark 1.3.5 below).

Let $\tilde{X}$ be the blowup of $X$ at $C$ and let $f : \tilde{X} \to \mathbb{P}^{n-1}$ be the morphism corresponding to $H_1, \ldots, H_n$.

The map $f$ is an elementary fibration in the sense of [4, Definition 3.1]; that is, it fits into
a commutative diagram of the form

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{i} & X \\
\downarrow & \searrow & \downarrow \\
S & \swarrow & Z \\
\end{array}
$$

The pullback of $f$ to $\tilde{X}$ is an elementary fibration $\tilde{f} : \tilde{X} \to S$ such that $\tilde{f} \circ i$ is an elementary fibration $i : X \to S$.
(currently with \( S = \mathbb{P}^{n-1}_T \)) in which:

- \( j \) is an open immersion with dense image in each fiber, and \( i \) is a closed immersion such that \( \tilde{X} = X \setminus Z \);
- \( \tilde{f} \) is smooth projective with fibers which are geometrically irreducible of dimension 1;
- \( g \) is finite étale and surjective.

By shrinking \( X \), we may further ensure that \( \deg(g) \geq 3 \).

To make \( \tilde{f} \) into a smooth curve fibration, we must force \( g \) to become a disjoint union of sections; this can be achieved by replacing \( S \) with a finite étale cover. More precisely, take any component of \( Z \) which does not map isomorphically to \( S \); this component is itself a finite étale cover of \( S \), and pulling back along it produces a fibration in which the inverse image of \( Z \) splits off a component which maps isomorphically to \( S \), so we may repeat the construction to achieve the desired result.

Finally, by Corollary 1.3.3, after replacing \( S \) with a suitable alteration, we get an extension of \( \tilde{f} \) to a stable curve fibration over some compactification \( \overline{S} \) of \( S \). By applying Proposition 1.3.2 we may ensure that \( \overline{S} \) is in fact a good compactification of \( S \); this completes the proof.

Remark 1.3.5. The construction of Corollary 1.3.4 has the following side effect which will be of use to us. For \( \tilde{f} \) the relative compactification of \( f \), there exists a section \( s \) of \( \tilde{f} \) whose image is the complement of \( U \) in another open subscheme \( U_1 \) which is also an elementary fibration over \( S \). The image of \( s \) is the intersection of \( U \) with one of the exceptional divisors of the blowup \( \tilde{Y} \to Y \); in particular, this image contracts to a point in \( Y \). For an application of this remark, see Lemma 2.2.4.

1.4. Moduli of coherent sheaves. We now introduce a different moduli stack that will be more closely related to crystals.

Hypothesis 1.4.1. Throughout §1.4, let \( f : Y \to B \) be a separated morphism of finite presentation of algebraic spaces over some base scheme \( S \).

Definition 1.4.2. Let \( \text{Coh}_{Y/B} \) denote the category in which:

- the objects are triples \((T, g, \mathcal{F})\) in which \( T \) is a scheme over \( S \), \( g : T \to B \) is a morphism over \( S \), and (writing \( Y_T := Y \times_B T \)) \( \mathcal{F} \) is a quasicoherent \( \mathcal{O}_{Y_T} \)-module of finite presentation which is flat over \( T \) and has support which is proper over \( T \);
- the morphisms \((T', g', \mathcal{F}') \to (T, g, \mathcal{F})\) consist of pairs \((h, \psi)\) in which \( h : T' \to T \) is a morphism of schemes over \( B \) and (writing \( h' : Y_{T'} \to Y_T \) for the base extension of \( h \) along \( f \)) \( \psi : (h')^* \mathcal{F} \to \mathcal{F}' \) is an isomorphism of \( \mathcal{O}_{Y_{T'}} \)-modules.

These form a stack over \( B \) via the functor \((T, g, \mathcal{F}) \mapsto (T, g)\).

Proposition 1.4.3. The category \( \text{Coh}_{Y/B} \) is an algebraic stack over \( S \).

Proof. See [61, Tag 09DS]. □

Proposition 1.4.4. The morphism \( \text{Coh}_{Y/B} \to B \) is quasiseparated and locally of finite presentation.

Proof. See [61, Tag 0DLZ]. Additional references, which impose more restrictive hypotheses but would still suffice for our purposes, are [43, Théorème 4.6.2.1] and [45, Theorem 2.1]. □
Lemma 1.4.5. Let $T$ be the spectrum of a valuation ring, let $\eta \in T$ be the generic point, and let $j : \eta \to T$ be the canonical inclusion. Fix a morphism $g : T \to B$ and an object of $\text{Coh}_{Y/B}$ of the form $(\eta, g \circ j, \mathcal{F})$. Then $j_*\mathcal{F}$ is the colimit of its coherent $T$-flat subsheaves.

Proof. We follow [61, Tag 0829]. Since $f$ is quasicompact and quasiseparated, we may reduce at once to the case where it is affine. In that case, write $T = \text{Spec}(R)$, $\eta = \text{Spec}(K)$, and $X_T = \text{Spec}(A)$; we must show that a finitely presented $A \otimes_R L$-module $M$ is the colimit of its finitely presented $R$-flat $A$-submodules. Since $A \otimes_R L$ is torsion-free as an $R$-module, every $R$-submodule of it is flat; the claim is therefore apparent. (Compare [61, Tag 0829].) \qed

Proposition 1.4.6. The map $\text{Coh}_{Y/B} \to B$ satisfies the existence part of the valuative criterion [61, Tag 0CLK].

Proof. As in [61, Tag 0DM0], this reduces at once to Lemma 1.4.5. \qed

Unless $f$ is finite, we cannot hope for $\text{Coh}_{Y/B} \to B$ to be quasicompact. The best we can do is cover $\text{Coh}_{Y/B}$ with open substacks which are themselves quasicompact over $B$.

Definition 1.4.7. Assume that $f$ is projective and $B$ is quasicompact, and let $\mathcal{L}$ be a line bundle on $Y$ which is very ample relative to $f$. For any object $(T, \mathcal{F}) \to \text{Coh}_{Y/B, n}$, we may define the associated Hilbert function

$$P : T \mapsto \mathbb{Q}[t], \quad P(x)(t) = \chi(Y \times_T x, \mathcal{L}_x^* (\mathcal{F} \otimes \mathcal{L}^{\otimes t}))$$

where $\iota_x : x \to T$ denotes the canonical inclusion. This function is locally constant [61, Tag 0DIZ].

For $P \in \mathbb{Q}[t]$, let $\text{Coh}_{Y/B}^{P, \mathcal{L}}$ be the substack of $\text{Coh}_{Y/B}$ consisting of those triples $(T, g, \mathcal{F})$ for which the Hilbert function of $\mathcal{F}$ (with respect to $\mathcal{L}$) is identically equal to $P$. As per [61, Tag 0DNF], $\text{Coh}_{Y/B}^{P, \mathcal{L}}$ is a closed-open substack of $\text{Coh}_{Y/B}$ and $\text{Coh}_{Y/B}$ is equal to the disjoint union of the $\text{Coh}_{Y/B}^{P, \mathcal{L}}$ over all $P$.

For a positive integer, let $\text{Coh}_{Y/B}^{P, \mathcal{L}, m}$ be the locally closed substack of $\text{Coh}_{Y/B}^{P, \mathcal{L}}$ consisting of those triples $(T, g, \mathcal{F})$ for which $f^*f_* (\mathcal{F} \otimes g^* \mathcal{L}^{\otimes m}) \to \mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is surjective and $R^i f_* (\mathcal{F} \otimes g^* \mathcal{L}^{\otimes m}) = 0$ for all $i > 0$. Note that $\text{Coh}_{Y/B}^{P, \mathcal{L}}$ is the union of the $\text{Coh}_{Y/B}^{P, \mathcal{L}, m}$ over all $m$.

Proposition 1.4.8. Assume that $f$ is projective and $B$ is quasicompact. Then for any $P, \mathcal{L}, m$ as in Definition 1.4.7, $\text{Coh}_{Y/B}^{P, \mathcal{L}, m}$ is quasicompact.

Proof. It suffices to produce a quasicompact algebraic space $W$ which surjects onto $\text{Coh}_{Y/B}^{P, \mathcal{L}, m}$. Let $Y \to \mathbb{P}_B^n$ be the projective embedding defined by $\mathcal{L}^{\otimes m}$. Let $P_m$ be the polynomial with $P_m(t) = P(m + t)$ and put $r = P_m(0)$; then each fiber of $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is globally generated by its $r$-dimensional space of global sections. We may thus take $W$ to be the Quot space $\text{Quot}^P_{\mathcal{O}_{\mathbb{P}^n_B/m \mathbb{P}^n_B/B}}$, which is proper over $B$ by [61, Tag 0DPA]. \qed

Proposition 1.4.9. Assume that $f$ is projective. Then for any $P, \mathcal{L}, m$ as in Definition 1.4.7, $\text{Coh}_{Y/B}^{P, \mathcal{L}, m} \to B$ is universally closed.

Proof. By Proposition 1.4.6, $\text{Coh}_{Y/B}^{P, \mathcal{L}, m} \to B$ satisfies the existence part of the valuative criterion. Since it is also quasicompact by Proposition 1.4.8, we may then apply [61, Tag 0CLW] to deduce that it is universally closed. \qed
2. COEFFICIENT OBJECTS

We review some relevant properties of coefficient objects and companions. Since we will be using terminology and notation from both [36] and [37], often with little comment, we recommend keeping those sources handy while reading.

2.1. COEFFICIENT OBJECTS AND ALGEBRAICITY.

**Definition 2.1.1.** By a coefficient object on $X$, we will mean an object of one of the categories $\text{Weil}(X) \otimes \overline{\mathbb{Q}}_\ell$, the category of lisse $\mathbb{Q}_\ell$-sheaves on $X$ for some prime $\ell \neq p$; or $\mathcal{F}\text{-Isoc}^{\dagger}(X) \otimes \overline{\mathbb{Q}}_p$, the category of overconvergent $F$-isocrystals on $X$ with coefficients in $\overline{\mathbb{Q}}_p$, in the sense of [36, Definition 9.2]. By the base coefficient field of a coefficient object, we mean the field $\mathbb{Q}_\ell$ in the former case and $\mathbb{Q}_p$ in the latter case. By the full coefficient field of a coefficient object, we mean the algebraic closure of the base coefficient field (without completion).

**Lemma 2.1.2.** Let $U$ be an open dense subscheme of $X$. For any category of coefficient objects, restriction from coefficient objects over $X$ to coefficient objects over $U$ is fully faithful.

*Proof.* See [37, Lemma 1.2.2]. □

**Definition 2.1.3.** We say that a coefficient object on $X$ is constant if it arises by pullback from $\text{Spec}(k)$. We refer to the operation of tensoring with a constant coefficient object of rank 1 as formation of a constant twist.

**Lemma 2.1.4.** For every coefficient object $E$ on $X$, there exists a constant twist of $E$ whose determinant is of finite order.

*Proof.* See [37, Lemma 1.1.3]. □

**Definition 2.1.5.** For $E$ a coefficient object on $X$ and $x \in X^\circ$, let $F_x$ denote the linearized Frobenius action on $E_x$, and let $P(E_x, T)$ denote the reverse characteristic polynomial (Fredholm determinant) of $F_x$ in the variable $T$. We say that $E$ is algebraic if $P(E_x, T) \in \mathbb{Q}[T]$ for all $x \in X^\circ$; in this case, we have $P(E_x, T) \in E[T]$ for some number field $E$ independent of $x$ [37, Theorem 3.3.2]. To specify such a number field, we may say that $E$ is $E$-algebraic.

**Lemma 2.1.6.** Let $E$ be a coefficient object on $X$. If $E$ is irreducible and $\det(E)$ is of finite order, then $E$ is algebraic.

*Proof.* See [37, Corollary 3.3.3]. □

**Remark 2.1.7.** Note that in a certain sense, the categories of coefficient objects on $X$ do not depend on the base field $k$; see [36, Definition 9.2] for discussion of this point in the crystalline case. For this reason, it will often be harmless for us to assume that $X$ is geometrically irreducible.

The following is [37, Corollary 3.3.5].

**Lemma 2.1.8.** Fix a category of coefficient objects and an embedding of $\overline{\mathbb{Q}}$ into the full coefficient field. Let $E$ be a number field within $\overline{\mathbb{Q}}$ and let $L_0$ be the completion of $E$ in the full coefficient field. Let $E$ be an $E$-algebraic coefficient object on $X$ (in the specified category) of rank $r$. Then there exists a finite extension $L_1$ of $L_0$, depending only on $L_0$ and $r$ (but not on $X$ or $E$ or $E$), for which $E$ can be realized as an object of $\text{Weil}(X) \otimes L_1$ (in the étale case) or $\mathcal{F}\text{-Isoc}^{\dagger}(X) \otimes L_1$ (in the crystalline case).
2.2. Étale coefficients on elementary fibrations. We next establish a certain rigidity property for étale coefficient objects on curve fibrations. A similar statement holds in the crystalline case, but we will not need it here. Before proceeding, we start with a motivating observation from complex geometry.

Remark 2.2.1. Let \( f : X \to S \) be a smooth \( n \)-pointed genus-\( g \) fibration of complex analytic spaces for some \( n, g \), and let \( \mathcal{E} \) be a local system on \( X \). Since the underlying morphism of topological spaces is a fiber bundle, the sheaves \( R^i f_* \mathcal{E} \) are again local systems.

Lemma 2.2.2. Let \( f : X \to S; s_1, \ldots, s_n : S \to X \) be a smooth \( n \)-pointed genus-\( g \) fibration of \( k \)-schemes with unpointed locus \( U \). Let \( \mathcal{E} \) be an étale coefficient object on \( U \) which is tame along \( X \setminus U \). Then the sheaves \( R^i f_{et,*} \mathcal{E} \) are étale coefficient objects on \( S \) for \( i \geq 0 \), and vanish for \( i > 2 \); moreover, the formation of these commutes with arbitrary base change on \( S \).

Proof. These statements reduce to the corresponding statements for a torsion étale sheaf on \( X \setminus \overline{U} \) of order prime to \( p \). In this context, the sheaves \( R^i f_{et,*} \mathcal{E} \) are constructible and vanish for \( i > 2 \). To check that they are lisse, using the proper base change theorem it suffices to check that the cohomology groups of \( E \) on geometric fibers have locally constant dimension. For \( i = 0 \), this follows from the tame specialization theorem [26]; this implies the case \( i = 2 \) by Poincaré duality. Given these cases, the case \( i = 1 \) follows from the local constancy of the Euler characteristic, which is implied by the Grothendieck-Ogg-Shafarevich formula. \( \square \)

Corollary 2.2.3. For \( f, s_1, \ldots, s_n, U, Z \) as in Lemma 2.2.2, let \( \mathcal{E}, \mathcal{F} \) be two étale coefficient objects on \( U \) which are tame along \( Z \).

(a) Suppose that there exists a geometric point \( \overline{s} \) of \( S \) such that over \( U_{\overline{s}} \), there exists a nonzero morphism \( \mathcal{E} \to \mathcal{F} \). Then the image of the natural contraction map \( \mathcal{E} \otimes f^{*} f_{et,*}(\mathcal{E}^\vee \otimes \mathcal{F}) \to \mathcal{F} \) is a nonzero subobject of \( \mathcal{F} \).

(b) Suppose that the restriction of \( \mathcal{E} \) to \( U_{\overline{s}} \) is absolutely irreducible and isomorphic to the restriction of \( \mathcal{F} \). Then \( f_{et,*}(\mathcal{E}^\vee \otimes \mathcal{F}) \) is of rank 1 and \( \mathcal{E} \otimes f^{*} f_{et,*}(\mathcal{E}^\vee \otimes \mathcal{F}) \to \mathcal{F} \) is an isomorphism.

Proof. By Lemma 2.2.2, \( R^0 f_{et,*}(\mathcal{E}^\vee \otimes \mathcal{F}) \) is an étale coefficient object on \( S \) whose formation commutes with base change from \( S \) to \( \overline{s} \). Similarly, forming the image of \( \mathcal{E} \otimes R^0 f_{et,*}(\mathcal{E}^\vee \otimes \mathcal{F}) \to \mathcal{F} \) commutes with base change from \( S \) to \( \overline{s} \), and the given hypotheses imply that over \( \overline{s} \) the image is nonzero. This proves (a); (b) is similar. \( \square \)

The following is a variant of [37, Lemma 3.1.1].

Lemma 2.2.4. For \( f, s_1, \ldots, s_n, U, Z \) as in Lemma 2.2.2, let \( \mathcal{E} \) be an absolutely irreducible étale coefficient object on \( U \) which is tame along \( Z \). Suppose in addition that \( f \) admits a section \( s : S \to U \) such that \( s^{*} \mathcal{E} \) is trivial. Then for each \( x \in S^0 \), the restriction of \( \mathcal{E} \) to \( f^{-1}(x) \) is also absolutely irreducible.

Proof. By Lemma 2.2.2, \( f_{et,*}(\mathcal{E}^\vee \otimes \mathcal{E}) \) is an étale coefficient object. Since it is also a subobject of the trivial coefficient object \( s^{*} \mathcal{E} \), it must itself be trivial. Consequently, it cannot have rank greater than 1, as otherwise \( \mathcal{E} \) itself would fail to be absolutely irreducible. \( \square \)

2.3. The companion relation for coefficients.

Definition 2.3.1. Fix coefficient objects \( \mathcal{E} \) and \( \mathcal{F} \) on \( X \), as well as an isomorphism \( \iota \) between the algebraic closures of \( \mathbb{Q} \) in the full coefficient fields of \( \mathcal{E} \) and \( \mathcal{F} \) (which will often not be
mentioned explicitly). We say that \( E \) and \( F \) are companions (with respect to \( \iota \)) if for each \( x \in X^o \), the coefficients of \( P(E_x, T) \) and \( P(F_x, T) \) are identified via \( \iota \); in particular, this can only occur if both \( E \) and \( F \) are algebraic. Given \( E \), the companion relation determines \( F \) up to semisimplification (see Lemma 2.3.2 below). In case \( F \in F\text{-Isoc}^\wedge(X) \otimes \overline{Q}_p \), we also say that \( F \) is a crystalline companion of \( E \).

**Lemma 2.3.2.** Let \( E \) and \( F \) be coefficient objects on \( X \) which are companions.

(a) If \( E \) is irreducible, then so is \( F \).

(b) If \( E \) is absolutely irreducible (meaning that for any finite extension \( k' \) of \( k \), the pullback of \( E \) to \( X_{k'} \) is irreducible), then so is \( F \).

(c) If \( F' \) is another coefficient object in \( F \) which is also a companion of \( E \), then \( F \) and \( F' \) have isomorphic semisimplifications.

*Proof.* Part (a) is [37, Theorem 3.3.1(a)]. Part (b) is a formal consequence of (a). Part (c) is [37, Theorem 3.2.2]. \( \square \)

**Lemma 2.3.3.** Let \( U \) be an open dense subscheme of \( X \). Let \( E \) be a coefficient object on \( U \). Let \( F \) be a coefficient object on \( X \) whose restriction to \( U \) is a companion of \( E \). Then \( E \) extends to a coefficient object on \( X \), and any such extension is a companion of \( F \).

*Proof.* By [37, Corollary 3.3.3], there exists an extension of \( E \) which is a companion of \( F \). By Lemma 2.1.2, this is the unique extension of \( E \) to \( X \). \( \square \)

**Remark 2.3.4.** Let \( f : X' \to X \) be a radicial morphism. Then for any fixed category of coefficient objects, pullback via \( f \) defines an equivalence of categories between the coefficients over \( X \) and the coefficients over \( X' \).

**Lemma 2.3.5.** Let \( E \) be a coefficient object on \( X \). Let \( f : X' \to X \) be a dominant morphism. If \( f^*E \) admits a crystalline companion, then so does \( E \).

*Proof.* By restriction to a rational multisection of \( f \), we may put ourselves in the position where \( f \) is the composition of an open immersion with dense image and an alteration. We may handle the two cases separately; for this, see [37, Corollary 3.2.6, Corollary 3.5.3]. \( \square \)

2.4. Tame and docile coefficients. Using the fact that the existence of companions can be checked after an alteration (Lemma 2.3.5), we will be able to limit our attention to \( p \)-adic coefficient objects of a relatively simple sort. We repeat here [37, Definition 1.4.1].

**Definition 2.4.1.** Let \( X \hookrightarrow \overline{X} \) be an open immersion with dense image. Let \( D \) be an irreducible divisor of \( \overline{X} \) with generic point \( \eta \).

- For \( \ell \) a prime not equal to \( p \), an object \( E \) of \( \text{Weil}(X) \otimes \overline{Q}_\ell \) is tame (resp. docile) along \( D \) if the action of the inertia group at \( \eta \) on \( E \) is tamely ramified (resp. tamely ramified and unipotent).
- An object \( E \) of \( F\text{-Isoc}^\wedge(X) \otimes \overline{Q}_p \) is tame (resp. docile) along \( D \) if \( E \) has \( \mathbb{Q} \)-unipotent monodromy in the sense of [60, Definition 1.3] (resp. unipotent monodromy in the sense of [29, Definition 4.4.2]) along \( D \).

**Lemma 2.4.2.** For any prime \( \ell \neq p \) and any object \( E \) of \( \text{Weil}(X) \otimes \overline{Q}_\ell \), there exists an alteration \( f : X' \to X \) such that \( X' \) admits a good compactification with respect to which \( f^*E \) is docile.
Proof. See [37, Proposition 1.4.6]. This also covers the case \( \ell = p \), but we will not need that here. \qed

2.5. Companions on curves. We next recall the critical results on the existence of companions on curves, as extracted from the work of L. Lafforgue [40] and Abe [1] on the global Langlands correspondence in positive characteristic.

**Theorem 2.5.1** (L. Lafforgue, Abe). For \( X \) of dimension 1, every coefficient object on \( X \) which is irreducible with determinant of finite order is uniformly algebraic and admits companions in all categories of coefficient objects, which are again irreducible with determinant of finite order.

**Proof.** See [37, Theorem 2.2.1] and references therein. \qed

Although we frame the following statement as a corollary of Theorem 2.5.1, it also incorporates significant intermediate results of V. Lafforgue, Deligne, Drinfeld, Abe–Esnault, and the author; see [37] for more detailed attributions.

**Corollary 2.5.2.** Parts (i)-(v) of Theorem 0.1.1 hold in general. Under the additional hypothesis \( \dim(X) = 1 \), part (vi) of Theorem 0.1.1 also holds.

**Proof.** The first assertion is [37, Theorem 0.4.1]. The second assertion is [37, Theorem 0.2.1]. \qed

We record some refinements of this statement which we will also need.

**Corollary 2.5.3.** For \( X \) of dimension 1, any algebraic coefficient object is \( E \)-algebraic for some number field \( E \), and admits companions in all categories of coefficient objects.

**Proof.** See [37, Theorem 2.2.1]. \qed

**Corollary 2.5.4.** For any tame (resp. docile) coefficient object on \( X \), its companions are also tame (resp. docile).

**Proof.** For the proof when \( \dim(X) = 1 \) (which is the only case we will use here), see [37, Corollary 2.4.3]. The general case follows from this case using [37, Lemma 1.4.9]. \qed

2.6. Newton polygons. The following is [37, Lemma-Definition 4.3.2]; it subsumes (and corrects the proof of) [19, Theorem 1.3.3(i,ii)]. We state these results because we need them as part of the construction of crystalline companions, but a posteriori we will get much more precise results (see §3).

**Lemma-Definition 2.6.1.** Fix a category \( \mathcal{C} \) of coefficient objects and a normalized \( p \)-adic valuation \( v_p \) of the algebraic closure of \( \mathbb{Q} \) in the full coefficient field of \( \mathcal{C} \). For \( \mathcal{E} \) an algebraic object of \( \mathcal{C} \), there exists a function \( x \mapsto N_x(\mathcal{E}) \) from \( X \) to the set of polygons in \( \mathbb{R}^2 \) with the following properties.

(a) For \( x \in X^0 \), \( N_x(\mathcal{E}) \) equals the lower convex hull of the set of points

\[
\left\{ \left( i, \frac{1}{[\kappa(x):k]} v_p(a_i) \right) : i = 0, \ldots, d \right\} \subset \mathbb{R}^2
\]

where \( P(\mathcal{E}_x, T) = \sum_{i=0}^d a_i T^i \in \overline{\mathbb{Q}}[T] \) with \( a_0 = 1 \).
(b) For \( x \in X \) with Zariski closure \( Z \), for all \( z \in Z \), \( N_z(E) \) lies on or above \( N_x(E) \) with the same right endpoint. In particular, the right endpoint is constant on each component of \( X \).

(c) In (b), the set \( U \) of \( z \in Z^o \) for which equality holds is nonempty.

(d) In (c), for each curve \( C \) in \( Z \), the inverse image of \( U \) in \( C \) either is empty or consists of all but finitely many closed points.

(e) The vertices of \( N_x(E) \) all belong to \( \mathbb{Z} \times \frac{1}{N}\mathbb{Z} \) for some positive integer \( N \) (which may depend on \( X \) and \( E \)).

Corollary 2.6.2. For \( E \) a coefficient object on \( X \), the function \( x \mapsto N_x(E) \) on \( X \) assumes only finitely many values. In particular, the slopes of \( N_x(E) \) can be uniformly (in \( x \)) bounded above and below.

Proof. By part (b) of Lemma-Definition 2.6.1, we obtain the second assertion by considering the Newton polygons at the generic points of the irreducible components of \( X \). Adding part (e), we obtain the first assertion. \( \square \)

3. NEWTON POLYGONS OF ISOCRYSTALS

We recall some further properties of Newton polygons on isocrystals.

**Hypothesis 3.0.1.** Throughout §3, we assume only that the field \( k \) is perfect of characteristic \( p \), not necessarily finite. We continue to take \( X \) to be a smooth separated scheme of finite type over \( k \).

3.1. Convergent isocrystals, Newton polygons, and slope filtrations.

**Definition 3.1.1.** For \( L \) an algebraic extension of \( \mathbb{Q}_p \), let \( \textbf{F-Isoc}(X) \otimes L \) denote the category of convergent \( F \)-isocrystals with coefficients in \( L \). This category contains \( \textbf{F-Isoc}^\dagger(X) \otimes L \) as a full subcategory [36, Theorem 5.3].

For \( x \in X \) (not necessarily closed), we may give an intrinsic definition of the Newton polygon \( N_x(E) \) using the Dieudonné-Manin classification [36, Definition 3.3].

In terms of this intrinsic definition of Newton polygons, Lemma-Definition 2.6.1 may be refined as follows.

**Lemma 3.1.2.** Let \( L \) be a finite extension of \( \mathbb{Q}_p \). For \( E \in \textbf{F-Isoc}(X) \otimes L \), the function \( x \mapsto N_x(E) \) has the following properties.

(a) For \( k \) finite and \( x = X = \text{Spec}(k) \), the construction agrees with Lemma-Definition 2.6.1 (a).

(b) For \( x \in X \) with Zariski closure \( Z \), for all \( z \in Z \), \( N_z(E) \) lies on or above \( N_x(E) \) with the same right endpoint. In particular, the right endpoint is constant on each component of \( X \).

(c) In (b), the set \( U \) of \( z \in Z \) for which equality holds is open and Zariski dense, and its complement is of pure codimension 1 in \( Z \).

(d) In (c), for each curve \( C \) in \( Z \), the inverse image of \( U \) in \( C \) either is empty or consists of all but finitely many closed points.

(e) The vertices of \( N_x(E) \) all belong to \( \mathbb{Z} \times \frac{1}{N}\mathbb{Z} \) for \( N = \text{rank}(E)!\left[ L : \mathbb{Q}_p \right] \).
3.2. Slope filtrations. While the Dieudonné-Manin classification does not extend to the case where $X$ is not a point, a weaker version of the statement does generalize as follows. By Lemma 3.1.2, if $X$ is irreducible, then the hypothesis on the constancy of the Newton polygon can always be enforced after restricting from $X$ to a suitable open dense subscheme.

**Proposition 3.2.1.** Let $L$ be a finite extension of $\mathbb{Q}_p$. For $E \in \mathbf{F-Isoc}(X) \otimes L$, suppose that the function $x \mapsto N_x(E)$ is constant. Let $\mu_1 < \cdots < \mu_l$ be the slopes of $N_x(E)$ for any $x \in X$. Then $E$ admits a filtration (the slope filtration)

$$0 = E_0 \subset \cdots \subset E_l = E$$

such that for $i = 1, \ldots, l$, for all $x \in X$, $N_x(E_i/E_{i-1})$ consists of the single slope $\mu_i$.

**Proof.** Apply [36, Corollary 4.2].

The individual steps of the slope filtration can in turn be interpreted as representations of profinite fundamental groups.

**Definition 3.2.2.** An object $E \in \mathbf{F-Isoc}(X) \otimes L$ is unit-root if for all $x \in X$, $N_x(E)$ has all slopes equal to 0. By Lemma 3.1.2, it suffices to check this at the generic point of each irreducible component of $X$.

**Proposition 3.2.3.** Suppose that $X$ is irreducible and let $\overline{x} \to X$ be a geometric point. Let $L$ be a finite extension of $\mathbb{Q}_p$. Then there is a functorial (in $X$) equivalence of categories between unit-root objects of $\mathbf{F-Isoc}(X) \otimes L$ and continuous representations of the profinite étale fundamental group $\pi_1(X, \overline{x})$ on finite-dimensional $L$-vector spaces.

**Proof.** See [36, Theorem 3.7].

3.3. The minimal slope theorem. While the inclusion functor $\mathbf{F-Isoc}^\dagger(X) \otimes L \to \mathbf{F-Isoc}(X) \otimes L$ is fully faithful, it does not in general reflect subobjects; in particular, the slope filtration given by Proposition 3.2.1 does not in general lift back to $\mathbf{F-Isoc}^\dagger(X) \otimes L$. In some sense, one expects rather the opposite; a recent result of Tsuzuki [62] addresses a question raised in [36, Remark 5.14] to this effect.

**Theorem 3.3.1** (Tsuzuki). Suppose that $X$ is irreducible of dimension 1. Let $\mathcal{E}, \mathcal{F}$ be irreducible objects in $\mathbf{F-Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$. Let $U$ be an open dense subset of $X$ on which the functions $x \mapsto N_x(\mathcal{E}), x \mapsto N_x(\mathcal{F})$ are constant (this set exists by Lemma 3.1.2). Let $\mathcal{E}_1, \mathcal{F}_1 \in \mathbf{F-Isoc}(U) \otimes \overline{\mathbb{Q}}_p$ be the first steps of the slope filtrations of $\mathcal{E}, \mathcal{F}$, respectively, according to Proposition 3.2.1.

(a) Both $\mathcal{E}_1$ and $\mathcal{F}_1$ are irreducible in $\mathbf{F-Isoc}(U) \otimes \overline{\mathbb{Q}}_p$.

(b) If $\mathcal{E}_1 \cong \mathcal{F}_1$ in $\mathbf{F-Isoc}(U) \otimes \overline{\mathbb{Q}}_p$, then $\mathcal{E} \cong \mathcal{F}$ in $\mathbf{F-Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$.

**Proof.** For (a), see [62, Proposition 5.8]. For (b), see [62, Theorem 1.3]. Alternatively, both statements can be deduced from [13, Theorem 1.1.3].
4. Vector bundles on curves

We next assemble some standard statements about vector bundles on curves, with an eye towards controlling the Harder-Narasimhan polygons of vector bundles arising from isocrystals.

**Hypothesis 4.0.1.** Throughout §4, let $C$ be a proper curve of genus $g$ over an arbitrary field $L$.

4.1. **Harder–Narasimhan polygons.** We recall some standard properties of vector bundles on curves, using the language of Harder–Narasimhan (HN) polygons.

**Definition 4.1.1.** We define the degree of a nonzero vector bundle on $C$ as the degree of its top exterior power, setting the degree of the zero bundle to be 0. Define the slope of a nonzero vector bundle $E$ on $C$, denoted $\mu(E)$, as the degree divided by the rank. We record some basic properties.

- If $F$ is a subbundle of $E$ of the same rank, then $\deg(F) \leq \deg(E)$. (This reduces to the case where the common rank is 1.)
- If $0 \to F \to E \to G \to 0$ is a short exact sequence of vector bundles, then $\deg(F) + \deg(G) = \deg(E)$ and $\rank(F) + \rank(E) = \rank(G)$. In particular, the respective inequalities
  $$\mu(F) < \mu(E), \quad \mu(F) = \mu(E), \quad \mu(F) > \mu(E)$$

  are equivalent to
  $$\mu(G) > \mu(E), \quad \mu(G) = \mu(E), \quad \mu(G) < \mu(E).$$

- For $E^\vee$ the dual bundle of $E$, we have $\deg(E^\vee) = -\deg(E)$ and (if $E$ is nonzero) $\mu(E^\vee) = -\mu(E)$.

**Definition 4.1.2.** Let $E$ be a nonzero vector bundle over $C$. We say that $E$ is *semistable* if $\mu(E)$ is maximal among the slopes of nonzero subbundles of $E$. We say that $E$ is *stable* if ($E$ is semistable and) $\mu(F) < \mu(E)$ for every nonzero proper subbundle $F$ of $E$.

By a standard argument (e.g., see [35, §3.4]), every vector bundle $E$ on $C$ admits a unique filtration

$$0 = E_0 \subset \cdots \subset E_l = E$$

by subbundles such that each quotient $E_i/E_{i-1}$ is a vector bundle which is semistable of some slope $\mu_i$ and $\mu_1 > \cdots > \mu_l$; this is called the *Harder–Narasimhan filtration*, or *HN filtration*, of $E$. The associated *Harder–Narasimhan polygon*, or *HN polygon*, of $E$ is the graph of the function $[0, \rank(E)] \to \mathbb{R}$ which interpolates linearly between the points $(\rank(E_i), \deg(E_i))$ for $i = 0, \ldots, l$.

**Remark 4.1.3.** A useful characterization of the HN polygon of $E$ is as the boundary of the upper convex hull of the set of points $(\rank(F), \deg(F)) \in \mathbb{R}^2$ as $F$ varies over all subobjects $F$ of $E$. The proof is easy; see for example [35, Lemma 3.4.15].

**Remark 4.1.4.** Observe that the dual bundle $E^\vee$ satisfies $\mu(E^\vee) = -\mu(E)$ and is (semi)stable if and only if $E$ is. Consequently, the slopes of the HN polygon of $E^\vee$ are the negation of the corresponding slopes of $E$. 

Lemma 4.1.5. Let $F$ be an algebraic closure of $L$. Then a vector bundle on $C$ is semistable if and only if its pullback to $C_L$ is semistable.

Proof. See [42, Proposition 3].

Lemma 4.1.6. Let $E_1, E_2$ be vector bundles on $C$. Suppose that every slope of the HN polygon of $E_1$ is strictly greater than every slope of the HN polygon of $E_2$. Then $\text{Hom}(E_1, E_2) = 0$.

Proof. Observe first that if $E_1$ and $E_2$ are semistable and $E_1 \to E_2$ is a nonzero morphism with image $F$, then $F$ is both a quotient of $E_1$ and a subobject of $E_2$, so $\mu(E_1) \leq \mu(F) \leq (E_2)$. In particular, if $\mu(E_1) > \mu(E_2)$, then $\text{Hom}(E_1, E_2) = 0$.

We use this as the base case of an induction on $\text{rank}(E_1) + \text{rank}(E_2)$; we may also assume that $E_1, E_2$ are both nonzero as else there is nothing to check. Away from the base case, one of $E_1$ and $E_2$ is nonzero and not semistable. If it is $E_1$, then let $F$ be the first step of its HN filtration; given any homomorphism $E_1 \to E_2$, the induction hypothesis implies first that the restriction to $F$ is zero, and second that the induced homomorphism $E_1/F \to E_2$ is zero. A similar argument applies if $E_2$ is not semistable.

Remark 4.1.7. By comparison with Lemma 4.1.6, one may ask whether if every slope of the HN polygon of $E_1$ is strictly less than every slope of the HN polygon of $E_2$, then $\text{Hom}(E_1, E_2) = 0$. This is false; for example, if $E$ is a semistable vector bundle on $C$ of positive slope, it is not necessarily true that $H^0(C, E) \neq 0$.

That said, using Riemann-Roch, one can prove some weaker statements in this direction. For example, suppose that $\text{rank}(E) = 1$. Then $\dim L H^0(C, E) \geq \text{deg}(E) - g + 1$, so if $\text{deg}(E) \geq g$ then $H^0(C, E) \neq 0$. See also Lemma 4.1.8 below.

Lemma 4.1.8. Let $E$ be a vector bundle on $C$ such that every slope of the HN polygon of $E$ is greater than $2g - 1$. Then $E$ is generated by global sections.

Proof. By Lemma 4.1.5, we may reduce to the case where $L$ is algebraically closed. By Serre duality, we have

$$\dim L H^0(C, E) - \dim L H^0(C, \Omega_{C/L} \otimes E^\vee) = \text{rank}(E)(\mu(E) + 1 - g),$$

the case of rank 1 being handled by Remark 4.1.7. Since $\Omega_{C/L} \otimes E^\vee$ has all slopes less than -1, Lemma 4.1.6 implies that $H^0(C, \Omega_{C/L} \otimes E^\vee(P)) = 0$ for each $P \in C^\circ$ (here we use that $L$ is now algebraically closed). It follows that the trivial inequality

$$\dim L H^0(C, E) \leq \dim L H^0(C, E(-P)) + \text{rank}(E)$$

is an equality; this is only possible if the fiber of $E$ at $P$ is generated by global sections of $E$. This proves the claim.

Lemma 4.1.9. Let $0 \to E_1 \to E \to E_2 \to 0$ be an exact sequence of vector bundles on $C$.

(a) The HN polygon of $E$ lies on or above the union of the HN polygons of $E_1$ and $E_2$ (that is, the polygon in which each slope occurs with multiplicity equal to the sum of its multiplicities in the HN polygons of $E_1$ and $E_2$), with the same endpoint.

(b) Suppose that every slope of the HN polygon of $E_1$ is strictly less than every slope of the HN polygon of $E_2$. Then equality holds in (a) if and only if the sequence splits.

Proof. It is easy to see that if $E = E_1 \oplus E_2$, then the HN polygon of $E$ is equal to the union (see [35, Lemma 3.4.13]); one then deduces (a) using Remark 4.1.3 as in [35, Lemma 3.4.17].
Note that this also proves the “if” direction of (b). As for the “only if” direction of (b), suppose that equality holds, form the HN filtration of $E$, let $E'_1$ be the step of the filtration of rank equal to rank($E_2$), and put $E'_2 := E/E'_1$. By Lemma 4.1.6, the induced map $E'_1 \to E_1$ is zero, so we must have $E'_1 \cong E_2$; this gives the desired splitting. \qed

**Remark 4.1.10.** Given a curve $C$ over a scheme $S$ and a vector bundle $E$ over $C$, for each point $s \in S$ we may pull back $E$ to the fiber of $C$ over $s$ and compute the HN polygon of the resulting bundle; this defines a function from $S$ to the space of polygons which is upper semicontinuous [56, Theorem 3]. For example, taking $S = \text{Spec} \ W(k)$, this statement can be used to transfer bounds on HN polygons from the special fiber to the generic fiber. (Note that the right endpoint of the HN polygon is preserved under specialization, and so is locally constant.)

4.2. Gaps between slopes. In many cases, one can use extra structure on, or special properties of, a vector bundle on a curve to deduce constraints on the gaps between consecutive HN slopes, and thus on the HN polygon. We collect some statements of this form here; see §5.4 for a further argument in this vein.

**Definition 4.2.1.** For $E$ a nonzero vector bundle on $C$, we define the width of $E$, denoted width($E$), to be the maximum of $|\mu(E) - \mu_i|$ as $\mu_i$ varies over the slopes of the HN polygon of $E$. (This is not standard terminology.)

**Remark 4.2.2.** Let $E$ be a nonzero vector bundle on $C$ and consider a strictly increasing filtration $0 = E_0 \subset \cdots \subset E_l = E$ of $E$ by saturated subbundles; put $F_i := E_i/E_{i-1}$ for $i = 1, \ldots, l$. Then using Lemma 4.1.9, we see that

$$\text{width}(E) \leq \max\{\text{width}(F_i) : i \in \{1, \ldots, n\}\} + \max\{|\mu(F_i) - \mu(F_j)| : i, j \in \{1, \ldots, n\}\}.$$ 

In particular, if

$$0 \to E_1 \to E \to E_2 \to 0$$

is an exact sequence of nonzero vector bundles on $C$ and $\mu(E_1) = \mu(E_2)$, then width($E$) $\leq$ max\{width($E_1$), width($E_2$)\}.

**Remark 4.2.3.** Let $E$ be a vector bundle of degree $d$ and rank $r$ on $C$. Let $c$ be a real number with the property that no two consecutive slopes of the HN polygon of $E$ differ by more than $c$. Then

$$\frac{d}{r} = \mu(E) \geq \frac{1}{r} \sum_{j=0}^{r-1} (\mu_1 - jc) = \mu_1 - (r-1)c/2;$$

by this plus the corresponding argument for $E^\vee$, we see that width($E$) $\leq (r-1)c/2$. This remark is applicable in a variety of situations where one can bound the difference between consecutive slopes; see Lemma 4.2.4 and Lemma 4.2.5 for examples, and Lemma 5.4.6 for a closely related argument.

**Lemma 4.2.4.** For any indecomposable vector bundle $E$ on $C$, the differences between consecutive slopes of $E$ are bounded by $2g - 2$.

**Proof.** See [54, Proposition 2.1]. \qed
Lemma 4.2.5. Suppose that $L$ is of characteristic $p$ and let $\varphi_C$ denote the absolute Frobenius on $C$. Then for any semistable vector bundle $E$ on $C$, the differences between consecutive slopes of $\varphi_C^*E$ are bounded by $2g-2$.

Proof. See [57, Corollary 2].

Corollary 4.2.6. Suppose that $L$ is of characteristic $p$ and let $\varphi_C$ denote the absolute Frobenius on $C$. Let $E$ be a vector bundle of rank $r$ on $C$. Then

$$\operatorname{width}(\varphi_C^* E) \leq p \operatorname{width}(E) + (r-1)(g-1).$$

Proof. Equip $\varphi_C^* E$ with the $\varphi_C$-pullback of the HN filtration of $E$. For each $i$, by Remark 4.2.3 and Lemma 4.2.5 we have

$$\operatorname{width}(\varphi_C^* (E_i)/\varphi_C^* (E_{i-1})) = \operatorname{width}(\varphi_C^* (E_i/E_{i-1})) \leq (r-1)(g-1).$$

By Remark 4.2.2, this yields the desired result.

Lemma 4.2.7. Suppose that $L$ is of characteristic 0. Let $D$ be a finite union of closed points of $C$. Let $E$ be a vector bundle on $C$ admitting a logarithmic connection with singularities in $D$ for which all residues are nilpotent.

(a) If $L$ is of characteristic 0, then $\deg(E) = 0$.

(b) If $L$ is of characteristic $p$, then $\deg(E) \equiv 0 \pmod{p}$.

(c) Suppose that $L = k$. Let $n$ be a positive integer and let $C$ be a smooth proper curve over $W_n(k)$ with $C_k \cong C$. Let $D$ be a closed subscheme of $C$, smooth over $W_n(k)$, with $D_k = D$ as subschemes of $C$. Suppose that $E$ is the pullback to $C$ of a vector bundle on $C$ admitting a logarithmic connection with singularities in $D$ for which all residues are nilpotent. Then $\deg(E) \equiv 0 \pmod{p^n}$.

Proof. By taking the top exterior power, we may reduce all three cases to the setting where $\operatorname{rank} E = 1$; in this case, the connection has no singularities. By choosing a rational section $v$ of $E$, we may write the connection as $v \mapsto v \otimes \omega$ where $\omega$ is a meromorphic differential. By a local computation, the image in $L$ of the degree of $E$ equals the sum of the residues of $\omega$, which equals 0 by the residue theorem. This implies (a) and (b); we may similarly deduce (c) by repeating the computation on $C$.

Remark 4.2.8. Part (a) of Lemma 4.2.7 also follows from a formula of Ohtsuki [52, Theorem 3]: the existence of the connection implies that $E$ has vanishing first Chern class, and hence is of degree 0. For $D = \emptyset$, the corresponding statement dates back to Atiyah [5, Theorem 4].

More precisely, the Chern class formula of Ohtsuki involves a sum over poles of the connection, in which each summand involves the Chern polynomial evaluated at the residue. The Chern polynomial depends on its argument only up to semisimplification, so nilpotent residues contribute as if they were zero; the Chern class formula thus returns zero, just as it would in the case of an everywhere holomorphic connection.

What this shows is that some restriction on residues in Lemma 4.2.7 is essential to obtaining any meaningful restriction on $E$. In fact, without the residue restriction, every vector bundle on $C$ admits a logarithmic connection; see for example [8, Theorem 5.3].

On the other hand, we can still obtain a bound of the desired form if we insist that the exponents be not necessarily zero, but belong to some bounded interval such as $[0, 1)$. 

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Corollary 4.2.9. With notation as in Lemma 4.2.7(a), the slopes of the HN polygon of $E$ are bounded in absolute value by $(\text{rank}(E) - 1)(g - 1)$.

Proof. We may assume at once that $E$ is indecomposable. By Remark 4.2.3 and Lemma 4.2.4, $\text{width}(E) \leq (\text{rank}(E) - 1)(g - 1)$. By Lemma 4.2.7, this yields the desired result. □

Lemma 4.2.10. Fix an ample line bundle $O(1)$ on $C$. Let $E$ be a vector bundle on $C$. Then for every integer $n > \text{width}(E) - \mu(E(n)) + 2g - 1$, $E(n)$ is generated by global sections.

Proof. By definition, $\mu(E(n)) - \text{width}(E) + n$ is a lower bound on the slopes of $E(n)$. Since this is greater than $2g - 1$, Lemma 4.1.8 implies that $E$ is generated by global sections. □

Remark 4.2.11. In Definition 4.1.2, if $L$ is of characteristic 0, then using transcendental methods (specifically the Narasimhan–Seshadri theorem [48] relating stable vector bundles on a compact Riemann surface to irreducible unitary representations of the fundamental group), one sees that the tensor product of two semistable vector bundles on $C$ is again semistable (compare [27, Proposition 2.2] for the corresponding argument for symmetric powers). However, this fails in positive characteristic (unless one factor is a line bundle); the first counterexamples are due to Gieseker [24, Corollary 1]. There are various ways to control the damage caused by this fact; see Lemma 4.2.12 for a simple argument that suffices for our purposes.

Lemma 4.2.12. For any two vector bundles $E_1, E_2$ on $C$, $\text{width}(E_1 \otimes E_2) \leq \text{width}(E_1) + \text{width}(E_2) + 4g$.

Proof. By Lemma 4.1.5, we may assume that $L$ is algebraically closed, so that there exists an ample line bundle $O(1)$ of degree 1 on $C$. Put $n_i = \lfloor \text{width}(E_i) - \mu(E_i) + 2g \rfloor$. By Lemma 4.2.10, $E_i(n_i)$ is generated by global sections, as then is $(E_1 \otimes E_2)(n_1 + n_2)$; the latter bundle therefore has all slopes nonnegative. This gives the claimed upper bound on the slopes of $E_1 \otimes E_2$; the claimed lower bound follows by a similar argument applies to the duals of $E_1, E_2$. □

5. Uniformities for isocrystals on curves

We next combine the previous discussion of vector bundles on curves with the general theory of isocrystals in order to derive a key uniformity property of isocrystals on curves.

5.1. Local lifting by formal schemes. In order to give a sufficiently concrete description of isocrystals for our work, we need to work with local lifts of varieties from characteristic $p$ to characteristic 0. We start with a version of the story which is not sufficient for our purposes, but is a natural starting point. (Our terminology here is not standard.)

Definition 5.1.1. By a smooth lift of $X$, we will mean a smooth formal scheme $P$ over $W(k)$ with $P_k \cong X$.

Lemma 5.1.2. Suppose that $X$ is affine. Then $X$ admits a smooth lift, which is unique up to noncanonical isomorphism.

Proof. This may be obtained by the method of Elkik [20], or more precisely by a result of Arabia [3, Théorème 3.3.2]. □
Lemma 5.1.3. Let \( \overline{f} : X' \to X \) be an étale morphism and let \( P \) be a smooth lift of \( X \). Then \( \overline{f} \) lifts functorially to an étale morphism \( f : P' \to P \) of formal schemes, where \( P' \) is a certain smooth lift of \( X' \) (determined by \( P \) and \( \overline{f} \)).

Proof. This is a consequence of the henselian property of the pair \((W(k), pW(k))\). See for example [23, Theorem 5.5.7], which is written in the more general context of almost commutative algebra, but is nonetheless a good reference for this point. \( \square \)

Definition 5.1.4. Let \( P \) be a smooth lift of \( X \). A Frobenius lift on \( P \) is a morphism \( \sigma : P \to P \) which acts on \( W(k) \) via the Witt vector Frobenius and lifts the absolute Frobenius morphism on \( X \).

The following construction gives a particular class of Frobenius lifts.

Definition 5.1.5. Let \((X, Z)\) be a smooth pair over \( k \). A smooth chart for \((X, Z)\) is a sequence \( t_1, \ldots, t_n \) of elements of \( \mathcal{O}_X(X) \) such that the induced morphism \( \overline{f} : X \to \mathbb{A}^n_k \) is étale and there exists \( m \in \{0, \ldots, n\} \) for which the zero loci of \( t_1, \ldots, t_m \) on \( X \) are the irreducible components of \( Z \) (and in particular are reduced).

Lemma 5.1.6. Let \((X, Z)\) be a smooth pair over \( k \). Then for each \( x \in X \), there exist an open subscheme \( U \) of \( X \) containing \( x \) and a smooth chart for \((U, Z \cap U)\).

Proof. By replacing \( x \) with a specialization, we may assume that \( x \in X^o \). Since \( X \) is smooth, it satisfies the Jacobian criterion; we can thus find elements \( \overline{t}_1, \ldots, \overline{t}_n \in \mathcal{O}_{X,x} \) such that \( \partial \overline{t}_1, \ldots, \partial \overline{t}_n \) form a basis of \( \Omega_{X/k,x} \) over \( \mathcal{O}_{X,x} \). By adjusting the choice of coordinates, we may further ensure that \( Z \) is cut out locally at \( x \) by \( \overline{t}_1 \cdots \overline{t}_m \). Choose an open affine neighborhood \( U \) of \( x \) in \( X \) omitting every irreducible component of \( Z \) not passing through \( x \). Then \( \overline{t}_1, \ldots, \overline{t}_n \) form a smooth chart for \((U, Z \cap U)\). \( \square \)

Definition 5.1.7. Let \((X, Z)\) be a smooth pair and let \( \overline{t}_1, \ldots, \overline{t}_n \) be a smooth chart for \((X, Z)\). Let \( P_0 \) be the formal completion of \( \text{Spec} \, W(k)[t_1, \ldots, t_n] \) along the zero locus of \( \overline{p} \).

By Lemma 5.1.3, there exists a unique smooth affine formal scheme \( P \) over \( W(k) \) equipped with an étale morphism \( f : P \to P_0 \) lifting \( \overline{f} \); we refer to \((P, t_1, \ldots, t_n)\) as the lifted smooth chart associated to the original smooth chart.

Let \( \sigma_0 : P_0 \to P_0 \) be the Frobenius lift for which \( \sigma_0^p(t_i) = t_i^p \) for \( i = 1, \ldots, n \). By the functoriality aspect of Lemma 5.1.3, there exists a unique Frobenius lift \( \sigma \) on \( P \) making the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{\sigma} & P \\
\downarrow f & & \downarrow f \\
P_0 & \xrightarrow{\sigma_0} & P_0 \\
\end{array}
\]

commute. We call \( \sigma \) the associated Frobenius lift of the lifted smooth chart.

Definition 5.1.8. For \( P \) a smooth lift of \( X \), denote by \( P_K \) the Raynaud generic fiber of \( P \); this is a rigid analytic space whose points correspond to formal subschemes of \( P \) which are integral and finite flat over \( W(k) \). In particular, there is a specialization map taking any such point to the intersection of \( X \) with the corresponding formal subscheme. For \( S \subseteq P \), let \( |S|_P \) denote the inverse image of \( S \) under the specialization map; this set is called the tube of \( S \) within \( P_K \).
5.2. **Weak formal schemes and dagger spaces.** We then upgrade the previous discussion to handle overconvergent isocrystals, by introducing certain analogues of formal schemes and rigid analytic spaces which directly incorporate the notion of overconvergence.

**Definition 5.2.1.** A ring $R$ is weakly complete with respect to an ideal $I$ if it is $I$-adically separated and, for any positive real numbers $a, b$ and any elements $x_1, \ldots, x_n \in R$, any infinite sum of the form

$$\sum_{i_1, \ldots, i_n = 0}^\infty a_{i_1 \ldots i_n} x_1^{i_1} \cdots x_n^{i_n} (a_{i_1 \ldots i_n} \in I^{[a(i_1 + \cdots + i_n) - b]})$$

converges in $R$. (By contrast, if $R$ is complete with respect to $I$, then the sum converges under the weaker condition that $a_{i_1 \ldots i_n} \in I^{f(i_1, \ldots, i_n)}$ where $f(i_1, \ldots, i_n) \to \infty$ as $i_1 + \cdots + i_n \to \infty$.) By replacing complete rings with weakly complete rings, we obtain Meredith’s concept of a weak formal scheme [47]; there is an obvious forgetful functor from weak formal schemes to formal schemes.

By a dagger lift of $X$, we will mean a smooth weak formal scheme $P^\dagger$ over $\text{Spf } W(k)$ with $P_k^\dagger \cong X$. We write $P$ for the underlying formal scheme of $P^\dagger$.

**Lemma 5.2.2.** Let $P^\dagger$ be a smooth weak formal scheme over $\text{Spf } W(k)$. Then $P^\dagger[p^{-1}]$ is noetherian and excellent. (The same is true of $P^\dagger$, but we will not need this fact.)

**Proof.** This reduces at once to the case where $P^\dagger$ is the weak completion of an affine space. Since $K$ is of characteristic 0, this case is an easy consequence of the Nullstellensatz for dagger algebras (see [25, §1.4]) plus the weak Jacobian criterion in the form of [46, Theorem 102]. □

**Lemma 5.2.3.** For $P^\dagger$ a smooth weak formal scheme over $\text{Spf } W(k)$, the morphism $P \to P^\dagger$ is faithfully flat.

**Proof.** Since we have surjectivity on points, it suffices to check flatness of the morphism on coordinate rings. This morphism is the direct limit of a family of morphisms, each of which corresponds to an open immersion of affinoid spaces over $K$ and so is flat [9, Corollary 7.3.2/6]. □

**Lemma 5.2.4.** Let $\overline{f} : X' \to X$ be an étale morphism and let $P^\dagger$ be a dagger lift of $X$. Then $\overline{f}$ lifts functorially to an étale morphism $f^\dagger : P^\dagger \to P^\dagger$ of weak formal schemes, where $P^\dagger$ is a certain dagger lift of $X'$ (determined by $P^\dagger$ and $\overline{f}$).

**Proof.** Note that if $R$ is weakly complete with respect to $I$, then the pair $(R, I)$ is henselian. We may thus argue as in the proof of Lemma 5.1.3. □

**Definition 5.2.5.** With notation as in Definition 5.1.7, let $P^\dagger_0$ be the weak formal completion of $\text{Spec } W(k)[t_1, \ldots, t_n]$ along the zero locus of $p$. By Lemma 5.2.4, $f$ descends uniquely to an étale morphism $f^\dagger : P^\dagger \to P^\dagger_0$ of weak formal schemes, and $\sigma$ descends to a morphism $\sigma^\dagger : P^\dagger \to P^\dagger$. We refer to $(P^\dagger, t_1, \ldots, t_n)$ as the lifted dagger chart associated to the original smooth chart.

**Definition 5.2.6.** For $P^\dagger$ a dagger lift of $X$, we may again define the generic fiber $P^\dagger_K$ as a locally $G$-ringed space with the same underlying $G$-topological space as $P_K$, but with a modified structure sheaf. The space $P^\dagger_K$ lives in the category of dagger spaces of Grosse-Klönne [25]; roughly, these are built in a fashion analogous to rigid analytic spaces, but
with the role of standard Tate algebras (i.e., the coordinate rings of generic fibers of formal completions of affine spaces over $W(k)$) being played by their overconvergent analogues (in which the formal completions become weak formal completions).

5.3. Relative GAGA for rigid and dagger spaces. It is well known that Serre’s GAGA theorem for varieties over $\mathbb{C}$ [55] has an analogue over a nonarchimedean field, in which complex analytic spaces are replaced by rigid analytic spaces. For completeness, we give a somewhat more general statement than the one we actually need. (Here we only use the fact that $K$ is a nonarchimedean field; the discreteness of the valuation plays no role.)

Proposition 5.3.1. Let $C$ denote either the category of rigid analytic spaces over $K$ or the category of dagger spaces over $K$. Suppose that $S$ is an affinoid space in $C$ and put $S_0 := \text{Spec} \mathcal{O}(S)$. Let $f_0 : Y_0 \to S_0$ be a proper morphism and let $f : Y \to S$ be the pullback of $f_0$ to $S$ along the natural morphism $S \to S_0$ of locally $G$-ringed spaces.

(a) The morphisms $S \to S_0, Y \to Y_0$ are flat and every closed point has a unique preimage.

(b) Pullback along the induced morphism $Y \to Y_0$ defines an equivalence of categories between coherent sheaves on $Y_0$ and on $Y$.

(c) Let $\mathcal{E}_0$ be a coherent sheaf on $Y_0$ and let $\mathcal{E}$ be the pullback of $\mathcal{E}_0$ to $Y$. Then the natural morphisms $R^i f_0^* \mathcal{E}_0 \to R^i f_* \mathcal{E}$ of sheaves on $S$ are isomorphisms for all $i$.

Proof. To prove (a), we need only treat the case $S \to S_0$. For this, see [25, Theorem 1.7] for the dagger case, and references therein for the rigid-analytic case.

We next skip to (c). Using Chow’s lemma as in the complex analytic case, we may reduce to the case where $f$ is projective, and then further to the case where $Y_0 = \mathbb{P}^n S_0$. By (a), pullback of coherent sheaves along $Y \to Y_0$ is an exact functor; we are thus free to make homological reductions. Using the relative ampleness of $\mathcal{O}(1)$ with respect to $\mathbb{P}^n S_0 \to S_0$, we may reduce to the cases where $\mathcal{E} = \mathcal{O}(m)$ for $m \in \mathbb{Z}$. These cases amount to the fact that the morphism

$$\mathcal{O}(S_0)[T_1^+, \ldots, T_n^+] \to \mathcal{O}(S_0)\langle T_1^+, \ldots, T_n^+ \rangle$$

in the rigid case, and the morphism

$$\mathcal{O}(S_0)[T_1^+, \ldots, T_n^+] \to \mathcal{O}(S_0)\langle T_1^+, \ldots, T_n^+ \rangle$$

in the dagger case, induces isomorphisms of associated graded rings for the grading by homogeneous degree. (This is essentially the method of Serre; see the first paragraph of [11, Example 3.2.6] for another approach that directly handles the case where $f_0$ is proper, without having to add Chow’s lemma.)

To prove (b), using (a) and (c) we know that the pullback functor is fully faithful. To establish essential surjectivity, we may again assume that $Y_0 = \mathbb{P}^n S_0$; again using (c), it suffices to check that for every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n S_0$, there exists an integer $m$ such that $\mathcal{F}(m)$ is generated by global sections. In the rigid-analytic case, we may appeal to [11, Theorem 3.2.4] to deduce this immediately. In the dagger case, we may make the corresponding argument after we note that as in Kiehl’s theorem, one knows that the higher direct images of a coherent sheaf along a proper morphism of dagger spaces are again coherent [25, Theorem 3.5].

Remark 5.3.2. Proposition 5.3.1, specialized to the case where $S$ is a point, includes the usual GAGA theorem for rigid analytic spaces over $K$, which has been known for some time (see the discussion in [11, Example 3.2.6]). It also includes GAGA for dagger spaces over $K$,
but this is not new either because the category of proper dagger spaces over \( K \) is equivalent to the category of proper rigid analytic spaces over \( K \) [25, Theorem 2.27].

As an application, we record a description of tame overconvergent isocrystals in the case where \( X \) has a liftable smooth compactification.

**Definition 5.3.3.** By a smooth lift of a smooth pair \((X, Z)\) over \( k \), we will mean a smooth pair \((\overline{X}, \overline{Z})\) of schemes (not formal schemes) over \( W(k) \) equipped with compatible identifications \( \overline{X}_k \cong X, \overline{Z}_k \cong Z \).

**Proposition 5.3.4.** Let \((\overline{X}, \overline{Z})\) be a smooth lift of a smooth pair \((X, Z)\) over \( k \) equipped with an identification \( X \cong \overline{X} \setminus \overline{Z} \). Let \( L \) be a finite extension of \( \mathbb{Q}_p \). Then there is a faithful functor from the category of docile objects of \( F\text{-Isoc}^\dagger(X) \otimes L \) to the category of vector bundles on \( \overline{X}_K \times_{\mathbb{Q}_p} L \) equipped with logarithmic (with respect to \( \mathfrak{Z}_K \times_{\mathbb{Q}_p} L \)) integrable connections.

**Proof.** We may reduce at once to the case \( L = \mathbb{Q}_p \). By Proposition 5.3.1, it suffices to prove the corresponding assertion with \((\overline{X}, \overline{Z})\) replaced by its \( p \)-adic formal completion. In that case, we first pass from docile objects of \( F\text{-Isoc}^\dagger(X) \) to the category of docile overconvergent isocrystals on \( X \) (without Frobenius structure), then apply [29, Theorem 6.4.1] to obtain a fully faithful functor from the latter category to vector bundles on \( \overline{X}_K \) equipped with logarithmic (with respect to \( \mathfrak{Z}_K \)) integrable connections.

**Remark 5.3.5.** In Proposition 5.3.4, one can get an equivalence of categories if \((\overline{X}, \overline{Z})\) admits a Frobenius lift (in which case the vector bundles also carry an action of this lift compatible with the connection); however, in practice such a lift almost never exists.

5.4. **Isocrystals on curves and slopes.** We now specialize to curves and combine with the theory of vector bundles to obtain the crucial uniformity. In passing, we mention that some related results have been obtained by Esnault–Shiho [22] and Bhatt–Lurie [7] from a somewhat different point of view.

**Hypothesis 5.4.1.** Throughout §5.4, we relax our running assumption and allow \( k \) to be an arbitrary perfect field. Let \((\overline{X}, \overline{Z})\) be a smooth pair in which \( \overline{X} \) is a proper curve over \( k \), and put \( X := \overline{X} \setminus \overline{Z} \). Let \( \eta \) be the generic point of \( X \). Let \( q \) be the genus of \( \overline{X} \) and let \( m \) be the \( k \)-length of \( Z \). Let \( \varphi : X \to X \) denote the absolute Frobenius morphism on \( X \).

Fix a smooth lift \((\overline{X}, \overline{Z})\) of \((X, Z)\) (which exists because of the smoothness of the moduli stack of curves; see Proposition 1.2.5) and put \( \mathfrak{F} := \overline{X} \setminus \overline{Z} \). Fix a finite extension \( L \) of \( \mathbb{Q}_p \) of degree \( d \) and an object \( \mathcal{E} \in F\text{-Isoc}^\dagger(X) \otimes L \) of rank \( r \) which is docile along \( Z \). As per Proposition 5.3.4, we realize \( \mathcal{E} \) as a vector bundle on \( \overline{X}_K \) equipped with a logarithmic integrable connection with nilpotent residues and a compatible action of \( L \).

**Definition 5.4.2.** Denote by \( W(\eta) \) the completed local ring of \( \overline{X} \) at \( \eta \); this is not the ring of Witt vectors over \( \eta \), but rather a Cohen ring with residue field \( \eta \). Let \( \mathcal{E}_\eta \) denote the pullback of \( \mathcal{E} \) to \( \text{Spec} W(\eta)[p^{-1}] \); this is a finite-dimensional vector space over \( W(\eta)[p^{-1}] \) equipped with a connection and a \( \mathbb{Q}_p \)-linear action of \( L \). Moreover, for any Frobenius lift \( \varphi \) on \( W(\eta) \), \( \mathcal{E}_\eta \) admits a semilinear \( \varphi \)-action on \( \mathcal{E}_\eta \) compatible with the connection and the \( L \)-action.

By a lattice in \( \mathcal{E} \), we will mean a vector bundle \( \mathfrak{E} \) on \( \overline{X} \) equipped with an isomorphism of \( \mathcal{E} \) with the pullback of \( \mathfrak{E} \) to \( \mathfrak{F} \). There is a pullback functor from lattices in \( \mathcal{E} \) with lattices in \( \mathcal{E}_\eta \). There is also a pullback functor from lattices in \( \mathcal{E} \) to vector bundles on \( X \), which we
call the reduction functor. Since $E$ has degree 0 by Lemma 4.2.7, any reduction of $E$ also has degree 0 by Remark 4.1.10.

**Definition 5.4.3.** A lattice $E$ in $E$ is crystalline if it is stable under the connection and the action of $o_L$, and if the corresponding lattice in $E_\eta$ is stable under the action of Frobenius. The latter condition is independent of the choice of a Frobenius lift on $W(\eta)$; we cannot assert it directly on $\mathcal{X}$ on account of Remark 5.3.5. However, the Frobenius structure does induce a well-defined action on the reduction of $E$.

**Lemma 5.4.4.** There exists a crystalline lattice in $E$ if and only if $N_\eta(E)$ has all slopes nonnegative.

**Proof.** If $E$ admits a crystalline lattice, then it is clear that $N_x(E)$ has all slopes nonnegative for each $x \in X^\circ$; by Lemma 2.6.1, $N_\eta(E)$ has all slopes nonnegative.

Conversely, suppose that $N_\eta(E)$ is nonnegative. To produce a crystalline lattice in $E$, it suffices to produce a lattice in $E_\eta$ stable under the connection, Frobenius, and $o_L$-action (this lattice will then extend to a reflexive sheaf on $\mathcal{X}$, which is a vector bundle because $\mathcal{X}$ is regular of dimension 2). To do this, start with any lattice, and then take its images under any finite combination of differentiation (with respect to some generic uniformizer), $\varphi$, and multiplication by scalars in $o_L$; we claim that the resulting images are uniformly bounded, and so generate a lattice stable under all of the operations. For the $o_L$-action this is obvious; for the $\varphi$-action, this is a consequence of the hypothesis on $N_\eta(E)$. Further details to follow. □

**Remark 5.4.5.** By Corollary 4.2.9, the slopes of the vector bundle $E$ on $\mathcal{X}_K$ are bounded in absolute value by $([L : \mathbb{Q}_p] \text{rank}(E) - 1)(g - 1)$. However, this does not directly imply anything about the slopes of a reduction of $E$, because the semicontinuity property of HN polygons goes in the wrong direction for this (see Remark 4.1.10). In fact, we do not claim anything about the slopes of an arbitrary reduction; rather, we will use crystalline lattices to produce specific lattices whose reductions we can control.

**Lemma 5.4.6.** Suppose that $N_\eta(E)$ has all slopes nonnegative. Then $E$ admits a crystalline lattice whose reduction has width at most

$$p^{dr} - 1 - dr(p - 1) \frac{dr - 1}{dr(p - 1)^2} (dr - 1)(g - 1) + \frac{dr - 1}{2} (2g - 2 + m).$$

Moreover, since the reduction has degree 0 (see Remark 5.4.5), this is also a bound on the absolute value of every HN slope of the reduction.

**Proof.** To simplify notation, we assume at once that $d = 1$. In light of Remark 4.2.2 and Lemma 4.2.7, we may also assume that $E$ is irreducible. In this case, we define a sequence of vector bundles $E_0, E_1, \ldots$ on $\mathcal{X}$ corresponding to crystalline lattices of $E$; show that this sequence must terminate; and show that the terminal value has the desired property. We start by applying Lemma 5.4.4 to construct $E_0$.

Given $E_i$, form the HN filtration of its reduction $E_{i,k}$. Suppose that there exists a pair of consecutive slopes $\mu_j, \mu_{j+1}$ of the HN polygon of $E_{i,k}$ such that

$$\mu_j - \mu_{j+1} > 2g - 2 + m, \quad p\mu_j - \mu_{j+1} > (r - 1)(g - 1).$$

By Lemma 4.1.6 plus the first inequality, the Kodaira-Spencer morphism $F_j \rightarrow E_{i,k}/F_j \otimes \Omega_{X/k}(Z)$ must vanish; thus $F_j$ is stable under the induced connection on $E_{i,k}$. Similarly, by
Lemma 4.2.5 plus the second inequality, the morphism $\varphi^* F_j \to \mathcal{E}_{i,k}/F_j$ must vanish, so $F_j$ is stable under the induced Frobenius action on $\mathcal{E}_{i,k}$. We may thus take $\mathcal{E}_{i+1}$ to be the inverse image of $F_j$ under the projection $\mathcal{E}_i \to \mathcal{E}_{i,k}$, and this will again be a crystalline lattice of $\mathcal{E}$.

By construction, there exists an exact sequence

$$0 \to \mathcal{E}_{i,k}/F_j \to \mathcal{E}_{i+1,k} \to F_j \to 0.$$  

By Lemma 4.1.9, the HN polygon of $\mathcal{E}_{i+1,k}$ is bounded below by the HN polygon of $\mathcal{E}_{i,k}$ with the same endpoints, and equality only holds if this sequence splits. In particular, the HN polygons of the $\mathcal{E}_{i,k}$ form a discrete, monotone, bounded sequence; this sequence must therefore either terminate or stabilize. If the sequence stabilizes, then the intersection of the $\mathcal{E}_j$ forms a proper subbundle of $\mathcal{E}$ stable under the Frobenius and the connection, contrary to hypothesis. We thus must have the desired termination.

It remains to deduce from termination the desired bound. Let $\mathcal{E}_i$ be the terminal value of the sequence. Let $\mu_1, \ldots, \mu_r$ be the slopes of the HN polygon of $\mathcal{E}_{i,k}$ listed with multiplicity. We then have

$$p \mu_i - \mu_{i+1} \leq (r - 1)(g - 1) \quad (\mu_i \geq (g - 1)(g - 1) - 2g + 2 - m)/(p - 1))$$

$$\mu_i - \mu_{i+1} \leq 2g - 2 + m \quad (\mu_i \leq (g - 1)(g - 1) - 2g + 2 - m)/(p - 1)).$$

Let $j$ be the number of slopes in this list which are greater than $(g - 1)(g - 1) - 2g + 2 - m)/(p - 1)$. Then by Lemma 4.2.7,

$$0 = \mu_1 + \cdots + \mu_r$$

$$\geq \sum_{h=1}^{j} (p^{h-1} \mu_1 - (p^{h-2} + \cdots + 1)(r - 1)(g - 1))$$

$$+ \sum_{h=j+1}^{r} (p^{j-1} \mu_1 - (p^{j-2} + \cdots + 1)(r - 1)(g - 1) - (h - j)(2g - 2 + m))$$

$$\geq r \mu_1 - \sum_{h=1}^{r} (p^{h-2} + \cdots + 1)(r - 1)(g - 1) - \sum_{h=1}^{r} (h - 1)(2g - 2 + m))$$

$$\geq r \mu_1 - \frac{p^{r-1} - 1 - r(p - 1)}{(p - 1)^2} (r - 1)(g - 1) - \frac{r(r - 1)}{2} (2g - 2 + m).$$

This gives an upper bound on $\mu_1$ of the desired form; repeating the argument for the dual of $\mathcal{E}$ gives a lower bound on $\mu_r$ of the desired form. □

**Remark 5.4.7.** Note that in Lemma 5.4.6, for fixed $p$ and $r'$, the slope bound is linear in $g$ and $m$. In this sense, the lemma is a refinement of the bound on jumping loci given in [37, Corollary 4.4.4]; that bound can be recovered from Lemma 5.4.6, but we will not give details here.

For our present purposes, there is no need to optimize the bound. We only use that it is a function of $p, r, g, m, L$.  

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6. Moduli of truncated crystals

In this section, we introduce a somewhat ad hoc notion of a “truncated crystal” on a smooth scheme (with logarithmic structure) over a perfect field. We then study some moduli stacks associated to this definition.

Hypothesis 6.0.1. Throughout §6, unless specified we assume only that \( k \) is perfect, not that it is finite.

6.1. Logarithmic schemes. In order to deal with boundaries of compactifications, we introduce log schemes. See [51] for a comprehensive treatment.

Definition 6.1.1. By a logarithmic (log) scheme, we will always mean a scheme equipped with a fine log structure in the sense of [28]. For example, this can (and generally will) be the log structure associated to a smooth pair as in [28, (1.5)(1)], which is indeed fine [28, Example (2.5)].

In more detail, for \( X \) a scheme, a pre-log structure on \( X \) consists of a sheaf of (commutative) monoids \( M \) on \( X_{et} \) together with a homomorphism \( \alpha : M \to \mathcal{O}_X \) with respect to multiplication on \( \mathcal{O}_X \); a log structure is a pre-log structure for which \( \alpha \) induces an isomorphism \( \alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times \); and a fine log structure is a log-structure for which \( M \) is cancellative and \( M/\alpha^{-1}(\mathcal{O}_X^\times) \) is locally finitely generated.

Definition 6.1.2. Let \((X,M)\) be a log scheme. As per [28, Definition 2.9], a chart for \( X \) is a homomorphism \( \beta : \mathbb{P}^1_X \to M \) where \( \mathbb{P}^1_X \) is the locally constant sheaf on \( X \) associated to \( P \), and \( \mathbb{P}^1_X/((\alpha \circ \beta)^{-1}(\mathcal{O}_X^\times)) \xrightarrow{\beta} M/\alpha^{-1}(\mathcal{O}_X^\times) \) is an isomorphism of sheaves of monoids on \( X_{et} \). (This last condition asserts that the adjunction from pre-log structures to log structures [28, (1.3)] promotes \( P \) to a fine log structure which is isomorphic to \( M \) via \( \beta \).) Such a chart always exists étale-locally on \( X \) [28, Lemma 2.10].

Definition 6.1.3. For \((Y,N) \to (X,M)\) a morphism of log schemes, let \( \Omega^1_{Y/X} \) denote the module of relative logarithmic differentials [28, (1.7)].

Remark 6.1.4. It is also possible to define logarithmic structures on algebraic stacks; see [53]. This will not be necessary for our purposes: the stacks we need to work with are moduli stacks of objects associated to logarithmic schemes, but no logarithmic structure on these stacks will be relevant here.

6.2. Interlude on Witt vectors. In order to get a better handle on crystals, we recall some facts about rings of finite \( p \)-typical Witt vectors.

Hypothesis 6.2.1. Throughout §6.2, fix a positive integer \( n \) and let \( R \) be a ring of characteristic \( p \).

Definition 6.2.2. Let \( W_n \) denote the endofunctor on rings given by taking \( p \)-typical Witt vectors of length \( n \). The projection \( W_n(R) \to W_1(R) = R \) admits a natural section \( \bullet : R \to W_n(R) \) at the level of multiplicative monoids. For \( \varphi : R \to R \) the Frobenius map (and the induced map on \( W_n(R) \)), there is a functorial (in \( R \)) additive homomorphism \( V : W_n(R) \to W_n(R) \), the Verschiebung map, with the property that \( \varphi \circ V = V \circ \varphi \) is multiplication by \( p \).
Remark 6.2.3. For $R$ perfect, $W_n(R)$ is the mod-$p^n$ truncation of the usual witt ring $W(R)$, which is $p$-adically separated and complete with $W(R)/(p) \cong R$ via the first projection. For general $R$, the structure of $W_n(R)$ is somewhat more complicated; for example, it is no longer generated over $\mathbb{Z}$ by the image of $[\bullet]$. However, if $R$ is of characteristic $p$, then it is true that $W_n(R)$ is the subring of $W_n(R)$ generated by $V^i([p^n]) = p^i[p^{p^n-1}]$ for $i = 0, \ldots, n-1$ and $r \in R$.

Lemma 6.2.4. Let $S$ be a ring and let $I$ be an ideal in $S$. Suppose that $I$ is nilpotent (that is, $I^n = 0$ for some positive integer $n$), $R/I$ is a noetherian ring, and $I/I^2$ is a finitely generated $R/I$-module. Then $R$ is noetherian.

Proof. We prove the “if” assertion, as the “only if” assertion will become clear during the argument. We apply Lemma 6.2.4 to the ring $S = W_n(R)$ and the ideal $I = \ker(W_n(R) \to R)$. We are given that $S/I \cong R$ is noetherian. For $i = 1, \ldots, n$, the image $V^i(W_n(R))$ of $V^i$ equals the kernel of $W_n(R) \to W_i(R)$. Since $V^i([x])V^i([y]) = p^iV^i([xy])$, the ideal $V^i(W_n(R))$ squares into $V^{i+1}(W_n(R))$, so all of these ideals are nilpotent. Since $I = V(W_n(R))$, it follows that $I$ is nilpotent.

To check that $I/I^2$ is finitely generated, by the previous paragraph we may assume $n = 2$. In this case, every element of $I$ can be written as $V([x])$ for some $x \in R$, and we have $[x]V([y]) = V([xy])$. Consequently, $I = I/I^2$ is isomorphic as an $R$-module to $R$ via Frobenius. Since $R$ is $F$-finite, it follows that $I$ is finitely generated.

Corollary 6.2.5. The ring $W_n(R)$ is noetherian if and only if $R$ is noetherian and $F$-finite (i.e., finite as an $R^p$-module). For example, this happens if $R$ is a localization of a finitely generated algebra over a perfect field.

Proof. We prove the “if” assertion, as the “only if” assertion will become clear during the argument. We apply Lemma 6.2.4 to the ring $S = W_n(R)$ and the ideal $I = \ker(W_n(R) \to R)$. We are given that $S/I \cong R$ is noetherian.

To check that $I/I^2$ is finitely generated, by the previous paragraph we may assume $n = 2$. In this case, every element of $I$ can be written as $V([x])$ for some $x \in R$, and we have $[x]V([y]) = V([xy])$. Consequently, $I = I/I^2$ is isomorphic as an $R$-module to $R$ via Frobenius. Since $R$ is $F$-finite, it follows that $I$ is finitely generated.

Definition 6.2.6. Let $S$ be a ring equipped with a homomorphism $\overline{f} : R \to S/pS$. Then there exists a natural homomorphism $f_n : W_n(R) \to S$ which makes the diagram

$$
\begin{array}{ccc}
W_n(R) & \xrightarrow{f_n} & S \\
\downarrow & & \downarrow \\
R & \xrightarrow{\overline{f}} & S/pS
\end{array}
$$

commute; namely, for $i = 0, \ldots, n-1$, $f_n$ carries $V^i([r^{p^n-1}])$ to $p^ix^{p^n-1}$ where $x \in S$ is any lift of $\overline{f}(r)$. In particular, any homomorphism $f : W_n(R) \to S$ lifting $\overline{f}$ must restrict to $f_n$.

6.3. Truncated crystals: local definition. We now give the promised ad hoc definition of truncated crystals which we will use to build genuine isocrystals via inverse limits (see §6.7). We first give a local description in coordinates.

Hypothesis 6.3.1. Throughout §6.3, fix a smooth affine scheme $S$ over $k$. Let $(X, Z)$ be a smooth pair over $k$ with $X$ affine (but not necessarily smooth) over $S$, and let $(P, t_1, \ldots, t_m)$
be a lifted smooth chart for \((X, Z)\) where the reductions of \(t_1, \ldots, t_d\) cut out \(Z\) within \(X\).

Fix a positive integer \(n\) and let \(P_n\) denote the reduction of \(P\) mod \(p^n\).

**Definition 6.3.2.** For a scheme \(T\) over \(k\), let \(T^{(n)}\) be a copy of \(T\) and let \(\varphi_n : T \to T^{(n)}\) denote the \(p^n\)-st power Frobenius. By Definition 6.2.6, the identification of \(P_n \times_{W_n(k)} k\) with \(X\) induces a morphism \(P_n \to W_n(X^{(n)})\) such that

\[
\begin{array}{ccc}
X & \rightarrow & P_n \\
\downarrow & & \downarrow \\
X^{(n)} & \rightarrow & W_n(X^{(n)})
\end{array}
\]

commutes.

Let \(P_{n,2}\) be the closure of the graph of the rational map \(P_n \times_{W_n(X^{(n)})} P_n \rightarrow \mathbb{G}_{m,k}^d\) given by \(\pi_1^*(t_1)/\pi_2^*(t_1), \ldots, \pi_1^*(t_d)/\pi_2^*(t_d)\). Let \(P_{n,3}\) be the closure of the graph of the rational map \(P_n \times_{W_n(X^{(n)})} P_n \times_{W_n(X^{(n)})} P_n \rightarrow \mathbb{G}_{m,k}^{2d}\) given by

\[
\pi_1^*(t_1)/\pi_3^*(t_1), \ldots, \pi_1^*(t_d)/\pi_3^*(t_d), \pi_2^*(t_1)/\pi_3^*(t_1), \ldots, \pi_2^*(t_d)/\pi_3^*(t_d)
\]

We then have projection maps

\[
\pi_1, \pi_2 : P_{n,2} \to P_n, \quad \pi_{12}, \pi_{13}, \pi_{23} : P_{n,3} \to P_{n,2}.
\]

Let \(\text{Crys}_{X_{\log}/S, n}\) be the category consisting of pairs \((T, \mathcal{F})\) in which \(T \in \text{Sch}_S\) and \(\mathcal{F}\) is a finitely generated \(\mathcal{O}\)-module on \(P_n \times_{W_n(S^{(n)})} W_n(T^{(n)})\), flat over \(W_n(T^{(n)})\) and with support proper over \(\text{Spec} W_n(T^{(n)})\), equipped with an isomorphism \(\iota : \pi_1^* \mathcal{F} \cong \pi_2^* \mathcal{F}\) on \(P_{n,2} \times_{W_n(S^{(n)})} W_n(T^{(n)})\) satisfying the cocycle condition

\[
\pi_{13}^*(\iota) = \pi_{23}^*(\iota) \circ \pi_{12}^*(\iota)
\]

on \(P_{n,3} \times_{W_n(S^{(n)})} W_n(T^{(n)})\). A morphism \((T', \mathcal{F}') \to (T, \mathcal{F})\) consists of a morphism \(f : T' \to T\) and a morphism \(\mathcal{F}' \to f^* \mathcal{F}\) compatible with \(\iota\).

**Remark 6.3.3.** We may think of an object of \(\text{Crys}_{X_{\log}/S, n}\) also as a tuple \((T, \mathcal{F}, F_1, F_2, \iota_1, \iota_2)\) in which \(T \in \text{Sch}_S\); \(\mathcal{F}\) is a finitely generated \(\mathcal{O}\)-module on \(P_n \times_{W_n(S^{(n)})} W_n(T^{(n)})\), flat over \(\mathbb{Z}/p^n\mathbb{Z}\); \(F_1, F_2\) are finitely generated \(\mathcal{O}\)-modules on \(P_{n,2} \times_{W_n(S^{(n)})} W_n(T^{(n)})\); and for \(j = 1, 2\), \(\iota_j : \pi_j^* \mathcal{F} \to F_j\) is a morphism of \(\mathcal{O}\)-modules on \(P_{n,2} \times_{W_n(S^{(n)})} W_n(T^{(n)})\). This tuple is then subject to the additional restrictions that \(\iota_1, \iota_2\) be isomorphisms and that the cocycle condition be satisfied.

Using this interpretation, we obtain a locally closed immersion

\[
(6.3.3.1) \quad \text{Crys}_{X_{\log}/S, n} \to \text{Coh}_{P_{n,2}/W_n(S^{(n)})} \times \text{Coh}_{P_{n,2}/W_n(S^{(n)})} \text{Coh}_{P_{n,2}/W_n(S^{(n)})}.
\]

of algebraic stacks over \(W_n(S^{(n)})\).

**Lemma 6.3.4.** Let \((P', t'_1, \ldots, t'_m)\) be a second lifted smooth chart for \((X, Z)\). Let \(\text{Crys}'_{X_{\log}/S, n}\) be the category defined by analogy with \(\text{Crys}_{X_{\log}/S, n}\) using the chart \((P', t'_1, \ldots, t'_m)\). Then there is a canonical equivalence of categories \(\text{Crys}_{X_{\log}/S, n} \cong \text{Crys}'_{X_{\log}/S, n}\); in particular, for a third choice of lifted smooth chart, the cocycle condition holds.
Proof. By Lemma 5.1.2, we can find an isomorphism $P \cong P'$ reducing to the identity modulo $p$, which we may use to define a functor $\text{Crys}_{\kappa^{\log}/S,n} \cong \text{Crys}'_{\kappa^{\log}/S,n}$. The claim is that there for two isomorphisms $f_1, f_2 : P \to P'$, we can find a natural isomorphism between the pullback functors $f_1^\ast$ and $f_2^\ast$ in a manner compatible with composition on either side. To see this, note that by Definition 6.2.6, we have a map $P_{n,2} \to P_{n,2}$ induced by $f_1$ and $f_2$; pulling $\iota$ back along this isomorphism gives rise to a natural isomorphism of the desired form, with compatibility with composition being guaranteed by the cocycle condition. □

6.4. **Truncated crystals: global definition.** We now globalize the previous definition to obtain truncated crystals without the requirement of local coordinates.

**Hypothesis 6.4.1.** For the remainder of §6, let $S$ be a smooth scheme of finite type over $k$, let $(X, Z)$ be a smooth pair over $k$, let $f : X \to S$ be a morphism, and let $n$ be a positive integer.

**Definition 6.4.2.** By Lemma 6.3.4, the category $\text{Crys}_{\kappa^{\log}/S,n}$ is canonically independent of the choice of a smooth lifted chart. We may thus consider these categories to form a stack for the étale topology on $S$; since smooth lifted charts always exist locally (Lemma 5.1.6) we may extend the definition of $\text{Crys}_{\kappa^{\log}/S,n}$ to the case where neither $X$ nor $S$ is required to be affine.

**Remark 6.4.3.** We view $\text{Crys}_{\kappa^{\log}/S,n}$ as a stack over $S$ via the functor $(T, F) \to T$. This requires some care: we have a commutative diagram

$$\begin{align*}
\text{Crys}_{\kappa^{\log}/S,n+1} & \longrightarrow \text{Crys}_{\kappa^{\log}/S,n} \\
\downarrow & \quad \downarrow \\
S & \bijection \phi : S
\end{align*}$$

in which the bottom horizontal arrow is Frobenius mapping right-to-left.

**Definition 6.4.4.** Given a commutative diagram

$$\begin{align*}
X' & \longrightarrow \quad X \\
\downarrow & \quad \downarrow \\
S' & \longrightarrow \quad S
\end{align*}$$

we obtain the pullback functor $h^\ast$ appearing in the diagram

$$\begin{align*}
\text{Crys}_{\kappa^{\log}/S,n} & \longrightarrow \quad \text{Crys}_{\kappa^{\log}/S',n} \\
\downarrow & \quad \downarrow \\
S & \bijection S'
\end{align*}$$

6.5. **Moduli stacks of truncated crystals.** We now discuss representability for the stacks $\text{Crys}_{\kappa^{\log}/S,n}$.

**Proposition 6.5.1.** The category $\text{Crys}_{\kappa^{\log}/S,n}$ is an algebraic stack locally of finite presentation over $S$. 

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Proof. It suffices to check this locally on $X$, so by Lemma 5.1.6 we may assume Hypothesis 6.3.1. Let us view $\text{Crys}^{X\log/S\!,n}$ as a stack over $W_n(S^{(n)})$ via the functor $(T, F) \mapsto W_n(T^{(n)})$. In light of Remark 6.3.3, we may apply Proposition 1.4.3 and Proposition 1.4.4 to deduce that $\text{Crys}^{X\log/S\!,n}$ is an algebraic stack over $W_n(S^{(n)})$ which is locally of finite presentation.

Now recall that the functor $T \mapsto W_n(T(n))$ is itself represented by a scheme of finite type over $S$. Hence by the previous paragraph, we may deduce the desired result. □

Lemma 6.5.2. Let $T$ be the spectrum of a valuation ring, let $\eta \in T$ be the generic point, and let $j : \eta \to T$ be the canonical inclusion. Fix a morphism $g : T \to B$ and an object of $\text{Crys}^{X\log/S\!,n}$ of the form $(\eta, F)$. Then $j_* F$ can be viewed as an ind-object in $\text{Crys}^{X\log/S\!,n}$ over $T$, with all morphisms injective.

Proof. The claim is local on $X$, so we may assume $f$ is affine. In that case, the proof of Lemma 1.4.5 adapts without incident. □

Proposition 6.5.3. The map $\text{Crys}^{X\log/S\!,n+1} \to \text{Crys}^{X\log/S\!,n}$ satisfies the existence part of the valuative criterion.

Proof. As in the proof of [61, Tag 0DM0], this follows formally from Lemma 6.5.2. □

Definition 6.5.4. Note that $\text{Crys}^{X\log/S\!,1} = \text{Coh}_{X/S}$ (independently of the choice of $Z$). Accordingly, for $P \in \mathbb{Q}[t]$, $L$ a line bundle on $X$ which is very ample relative to $f$, and $m$ a positive integer, we may define

$$\text{Crys}^{X\log/S\!,L,m} := \text{Crys}^{X\log/S\!,n} \times_{\text{Coh}_{X/S}} \text{Coh}^{P, L, m}_{X/S}.$$

Proposition 6.5.5. Suppose that $f$ is a stable curve fibration. Then for any $P, L, m$ as in Definition 6.5.4, the morphism $\text{Crys}^{P, L, m}_{X\log/S\!,n+1} \to \text{Crys}^{P, L, m}_{X\log/S\!,n}$ is universally closed and of finite type.

Proof. We may check the claim étale-locally on $S$. By Proposition 1.2.5, we may thus reduce to the case where $f$ lifts over a smooth formal lift of $S$; in which case we may deduce the finite type property directly from Proposition 1.4.8. The universally closed property follows from quasicompactness plus Proposition 6.5.3 using [61, Tag 0CLW]. □

Remark 6.5.6. While Proposition 6.5.5 will be sufficient for our purposes, we note that it should remain true assuming only that $f$ is projective. Since $f$ in general does not lift to characteristic 0, a proof of this may require de Jong’s theorem on semistable reduction via alterations [14, Theorem 8.2].

Proposition 6.5.7. Suppose that $f$ is a stable curve fibration. Then for any $P, L, m$ as in Definition 6.5.4, $\text{Crys}^{P, L, m}_{X\log/S\!,n}$ is universally closed and of finite type over $S$, and $\text{Crys}^{X\log/S\!,n}$ is the union of its open substacks $\text{Crys}^{P, L, m}_{X\log/S\!,n}$ over all positive integers $m$.

Proof. We proceed by induction on $n$. For $n = 1$, we identify $\text{Crys}^{P, L, m}_{X\log/S\!,n}$ with $\text{Coh}^{P, L, m}_{X/S}$ and invoke Proposition 1.4.8. Given the claim for some $n$, we factor the structure morphism $\text{Crys}^{P, L, m}_{X\log/S\!,n+1} \to S$ as per Remark 6.4.3 and note that the other three arrows in the diagram are finite type: the top horizontal arrow by Proposition 6.5.5, the right vertical arrow by the induction hypothesis, and the bottom horizontal arrow because $S$ is of finite type over a perfect field. □
6.6. **Frobenius structures on truncated crystals.** We now add Frobenius structures to the previous discussion. This is mostly straightforward except for the fact that absolute Frobenius does not act on $X$ relative to $S$.

**Definition 6.6.1.** Given a commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{h} & X \\
\downarrow & & \downarrow \\
S' & \xrightarrow{=} & S,
\end{array}
\]

by lifting $h$ locally to charts and arguing as in Lemma 6.3.4 to eliminate dependence on choices, we obtain the pullback functor $h^*$ appearing in the diagram

\[
\begin{array}{ccc}
\text{Crys}_{X}^{\log/S,n} & \xrightarrow{h^*} & \text{Crys}_{X'}^{\log/S',n} \\
\downarrow & & \downarrow \\
S & \xleftarrow{=} & S',
\end{array}
\]

In particular, for the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi_X} & X \\
\downarrow & & \downarrow \\
S & \xrightarrow{\varphi_S} & S
\end{array}
\]

we obtain a pullback functor which we denote simply by $\varphi^*$.

**Remark 6.6.2.** The pullback functor $\varphi^* : \text{Crys}_{X}^{\log/S,n} \to \text{Crys}_{X'}^{\log/S',n}$ carries $\text{Crys}_{X}^{P,L,n}$ into $\text{Crys}_{X'}^{P,\varphi^*L,M}$.

**Definition 6.6.3.** Let $\textbf{F-Crys}_{X}^{\log/S,n}$ be the category in which an object is an object $(T, F)$ of $\text{Crys}_{X}^{\log/S,n}$ together with a morphism $\Phi : \varphi^*F \to i^*F$ in $(\text{Crys}_{X}^{\log/S,n})_{T \times S', \varphi}$, where $i^*F$ is the pullback of $F$ along $i : T \times S', \varphi \to T$.

In addition to the natural first projection $\pi_1 : \textbf{F-Crys}_{X}^{\log/S,n} \to \text{Crys}_{X}^{\log/S,n}$ which forgets the extra data, there is a second projection $\pi_2 : \textbf{F-Crys}_{X}^{\log/S,n} \to \text{F-Crys}_{X}^{\log/S,n}$ carrying $(T, F)$ to $(T \times S', \varphi, \varphi^*F)$. Since $\textbf{F-Crys}_{X}^{\log/S,n}$ admits a natural projection to $\text{Crys}_{X}^{\log/S,n} \times \text{Crys}_{X}^{\log/S,n}$ via these two maps, we may view it as a stack over $S \times S$.

As in Remark 6.4.3, we have natural projection maps $\text{F-Crys}_{X}^{\log/S,n+1} \to \text{F-Crys}_{X}^{\log/S,n}$.

**Proposition 6.6.4.** The category $\textbf{F-Crys}_{X}^{\log/S,n}$ is an algebraic stack locally of finite presentation over $S \times S$.

**Proof.** This is immediate from Proposition 6.5.1.

**Lemma 6.6.5.** Let $T$ be the spectrum of a valuation ring, let $\eta \in T$ be the generic point, and let $j : \eta \to T$ be the canonical inclusion. Fix a morphism $g : T \to B$ and an object of $\textbf{F-Crys}_{X}^{\log/S,n}$ of the form $(\eta, F)$. Then $j_*F$ can be viewed as an ind-object in $\textbf{F-Crys}_{X}^{\log/S,n}$ over $T$, with all morphisms injective.

**Proof.** This is immediate from Lemma 6.5.2.
Proposition 6.6.6. The maps $\mathbf{F}$-$\text{Crys}^{P,L,m}_{X^{\log}/S,n+1} \rightarrow \mathbf{F}$-$\text{Crys}^{P,L,m}_{X^{\log}/S,n}$ and $\pi_1 : \mathbf{F}$-$\text{Crys}^{P,L,m}_{X^{\log}/S,n} \rightarrow S$ satisfy the existence part of the valuative criterion.

Proof. As in the proofs of [61, Tag 0DM0] and Proposition 6.5.3, this follows formally from Lemma 6.6.5. □

Definition 6.6.7. For $P, L, m$ as in Definition 6.5.4, define

$$\mathbf{F}$-$\text{Crys}^{P,L,m}_{X^{\log}/S,n} := \mathbf{F}$-$\text{Crys}^{P,L,m}_{X^{\log}/S,n} \times_{\pi_1, \text{Crys}^{P,L,m}_{X^{\log}/S,n}} \text{Crys}^{P,L,m}_{X^{\log}/S,n}.$$ 

Proposition 6.6.8. Suppose that $f$ is a stable curve fibration. Then for any $P, L, m$ as in Definition 6.5.4, $\mathbf{F}$-$\text{Crys}^{P,L,m}_{X^{\log}/S,n}$ is of finite type over $S \times S$.

Proof. This follows from Proposition 6.5.7 plus Remark 6.6.2. □

Proposition 6.6.9. Suppose that $f$ is a stable curve fibration. Then for any $P, L, m$ as in Definition 6.5.4, the map $\mathbf{F}$-$\text{Crys}^{P,L,m}_{X^{\log}/S,n+1} \rightarrow \mathbf{F}$-$\text{Crys}^{P,L,m}_{X^{\log}/S,n}$ is universally closed. In addition, $\pi_1 : \mathbf{F}$-$\text{Crys}^{P,L,m}_{X^{\log}/S,n} \rightarrow S$ is universally closed.

Proof. In both cases, the morphism in question is quasicompact by Proposition 6.6.8 and satisfies the existence part of the valuative criterion by Proposition 6.6.6, so by [61, Tag 0CLW] it is universally closed. □

6.7. Comparison with isocrystals. We finally reconcile our previous work with convergent isocrystals.

Lemma 6.7.1. There is a restriction functor from the category of convergent log-isocrystals on $(X, Z)$ to the isogeny category of $\lim \left\langle \text{Crys}_{X^{\log}/S,n} \right\rangle$ (inverting $p$ in Hom sets).

Proof. We use the interpretation of the category of convergent log-isocrystals as crystals on the log-crystalline topos of Shiho (see [59, Proposition 2.2.7] for the comparison with the definition we have been using up to now). Each smooth lifted chart $(P, t_1, \ldots, t_m)$ for an open subscheme of $X$ corresponds to an object of the log-crystalline site, on which we can evaluate a convergent log-isocrystal to get a vector bundle on $P \times_{\mathbb{Z}_p, \mathbb{Q}_p}$. By choosing a coherent (but not necessarily projective) lattice, we may descend to a coherent sheaf on $P$ which we may then restrict to $P_n$; we obtain the isomorphism $\iota$ (and its cocycle condition) from the rigidity property of crystals. □

Remark 6.7.2. We cannot directly invert the construction of Lemma 6.7.1 because the base spaces for our truncated crystals are too restrictive: we are modeling not convergent isocrystals, but rather “$p$-adically convergent isocrystals” in the sense of [49]. The difference between these and true convergent isocrystals is that the formal Taylor isomorphism defined by the connection is required to be convergent on a smaller region, corresponding to the closed disc $|T| \leq p^{-1}$ inside the open disc $|T| < 1$. Crucially, in the presence of Frobenius structures this difference goes away [49, Proposition 2.18]: this corresponds to Dwork’s observation (a/k/a “Dwork’s trick”) that a $p$-adic differential equation without singularities on an open unit disc in general only admits solutions on some disc by the $p$-adic Cauchy theorem, but in the presence of a Frobenius structure admits solutions on the whole disc [32, Corollary 17.2.2].

In light of the previous remark, we restrict our comparison statement from truncated crystals to isocrystals to the case where Frobenius structures are present.
Definition 6.7.3. Let $\text{F-Isoc}(X, Z)$ be the category of convergent log-isocrystals on $(X, Z)$ equipped with a Frobenius structure, which we insist is an isomorphism even over $Z$. Note that this enforces that the underlying log-isocrystal has nilpotent residues along $Z$: its residues form a finite multiset of a field of characteristic 0 stable under multiplication by $p$.

Let $\mathcal{C}$ be the full subcategory of the isogeny category of $\lim_n (\text{F-Crys}_{X^{\log}/S,n})_\mathbb{S}$ (inverting $p$ in Hom sets) consisting of objects for which the cokernel of Frobenius is killed by some power of $p$. Then Lemma 6.7.1 formally promotes to give a restriction functor $\text{F-Isoc}(X, Z) \to \mathcal{C}$.

Proposition 6.7.4. For the category $\mathcal{C}$ of Definition 6.7.3, there is a functor $\mathcal{C} \to \text{F-Isoc}(X \setminus Z, X)$ whose composition with the functor $\text{F-Isoc}(X, Z) \to \mathcal{C}$ from Lemma 6.7.1 is the usual restriction functor $\text{F-Isoc}(X, Z) \to \text{F-Isoc}(X \setminus Z, X)$.

Proof. Lemma 6.3.4 gives rise to a functor from $\mathcal{C}$ to the category of “$p$-adically convergent $F$-isocrystals on $(X, Z)$” in the sense of [49, Definition 2.7] (extrapolated from ordinary schemes to log schemes). As in [49, Proposition 2.18], these may then be promoted to true convergent $F$-isocrystals.

7. Companion points on moduli stacks of crystals

Using the existence of crystalline companions on curves, we study certain “companion points” on moduli stacks of crystals. The finiteness properties of moduli stacks of truncated crystals then lead to a crucial case of the existence of crystalline companions.

7.1. Setup.

Hypothesis 7.1.1. Throughout §7, let $S$ be a smooth, proper, geometrically irreducible scheme of finite type over $k$. Let $f : X \to S$ be a stable curve fibration (with $X$ smooth over $k$, as per our running assumption) with pointed locus $Z$. Let $\mathcal{E}$ be an étale coefficient object on $U := X \setminus Z$ which is docile along $Z$. Assume in addition that the restriction to $\mathcal{E}$ to each smooth fiber of $f$ is absolutely irreducible, and that the generic Newton polygon of $\mathcal{E}$ (Lemma-Definition 2.6.1) has least slope 0.

Definition 7.1.2. Let $V$ be the smooth locus of $f$ in $S$. For $x \in V^o$, by Theorem 2.5.1 there exists $\mathcal{F}_x \in \text{F-Isoc}^1(U \times_S x) \otimes \overline{\mathbb{Q}}_p$ which is a companion of $\mathcal{E}|_{U \times_S x}$; by Lemma 2.3.2(c), $\mathcal{F}_x$ is unique up to isomorphism.

By part (ii) of Theorem 0.1.1 (which is currently available in light of Corollary 2.5.2), $\mathcal{E}$ is $E$-algebraic for some number field $E$. By Lemma 2.1.8, this means that there exists a finite extension $L$ of $\mathbb{Q}_p$, chosen independently of $x$, for which $\mathcal{F}_x \in \text{F-Isoc}^1(U \times_S x) \otimes L$ for all $x \in V^o$. We fix such a choice hereafter.

Definition 7.1.3. Fix a finite extension $L$ of $\mathbb{Q}_p$ as in Definition 7.1.2. For $n$ a positive integer, let $\text{F-Crys}_{X^{\log}/S,n} \otimes \mathfrak{o}_L$ be the category of objects of $\text{F-Crys}_{X^{\log}/S,n} \otimes \mathfrak{o}_L$ equipped with a $\mathbb{Z}_p$-linear action of $\mathfrak{o}_L$. Similarly, for $P, \mathcal{L}, m$ as in Definition 6.5.4, let $\text{F-Crys}^{P,\mathcal{L},m}_{X^{\log}/S,n} \otimes \mathfrak{o}_L$ be the category of objects of $\text{F-Crys}^{P,\mathcal{L},m}_{X^{\log}/S,n} \otimes \mathfrak{o}_L$ equipped with a $\mathbb{Z}_p$-linear action of $\mathfrak{o}_L$.

For $x \in V^o$, a companion point (over $x$) on $\text{F-Crys}^{P,\mathcal{L},m}_{X^{\log}/S,n} \otimes \mathfrak{o}_L$ is an $x$-valued point of $\text{F-Crys}^{P,\mathcal{L},m}_{X^{\log}/S,n} \otimes \mathfrak{o}_L$ appearing in some sequence in $\lim_n (\text{F-Crys}_{X^{\log}/S,n})_x \otimes \mathfrak{o}_L$ mapping to $\mathcal{F}_x \in \text{F-Isoc}^1(U \times_S x) \otimes L$ as per Lemma 6.7.1.
Lemma 7.1.4. There exist a choice of $P, \mathcal{L}, m$ as in Definition 6.5.4 for which $F^{\text{Crys}}_{X_{\text{log}}/S, n} \otimes \mathcal{O}_L$ contains a companion point over every $x \in V^\circ$.

Proof. This follows from Lemma 5.4.6. □

7.2. Stabilization. Since Lemma 7.1.4 shows that $C^{\text{Crys}}_{P, \mathcal{L}, m, X_{\text{log}}/S, n} \otimes \mathcal{O}_L$ contains many companion points, our next move is to form the “stable Zariski closure” of the companion points therein. We then show that some common properties of companion points propagate to all points in this locus.

Hypothesis 7.2.1. Throughout §7.2, fix a finite extension $L$ of $\mathbb{Q}_p$ as in Definition 7.1.2 and a choice of $P, \mathcal{L}, m$ as in Lemma 7.1.4.

Definition 7.2.2. Define $M_n := \lim_{\longrightarrow} \text{image}(F^{\text{Crys}}_{X_{\text{log}}/S, n} \otimes \mathcal{O}_L \to F^{\text{Crys}}_{X_{\text{log}}/S, L, n} \otimes \mathcal{O}_L)$.

By Proposition 6.6.9, $M_n$ is a closed substack of $F^{\text{Crys}}_{X_{\text{log}}/S, n} \otimes \mathcal{O}_L$; by Proposition 6.6.8, it is of finite type over $S$. (In particular the limit stabilizes at some finite level by noetherianness, but this is not critical for our discussion.)

Let $M'_n$ be the Zariski closure of the companion points in $M_n$. By Lemma 7.1.4, the image of $M'_n$ in $S$ contains $V^\circ$.

Lemma 7.2.3. Let $x' \to V$ be a point valued in some finite extension $k'$ of $k$. Then for any $x'$-valued point of $M'_n$, the unit-root Frobenius trace is the same as for the corresponding companion point.

Proof. This holds for any companion point of $M'_n$ by definition. By the Chebotarev density theorem plus Proposition 3.2.3, for any morphism $C \to M'_n$ with $C$ a curve, if $C$ contains infinitely many closed points for which the assertion holds, then it holds for all closed points of $C$. Since by definition companion points are Zariski dense in $M'_n$, this implies the claim for all points valued in finite extensions of $k$. □

Using the minimal slope theorem, we now arrive at a decisive conclusion: every point of $M'_n$ valued in a finite extension of $k$ and lying over $V$ is a companion point. In fact, we prove something more precise.

Definition 7.2.4. Let $W$ be the subset of $x \in V^\circ$ for which the least generic slope of $N(\mathcal{F}_x)$ is 0. By hypothesis, $W$ is nonempty; by Lemma-Definition 2.6.1, for each curve $C$ in $S$, $C \cap W$ is either empty or contains all but finitely many points of $C^\circ$. In particular, $W$ is Zariski dense in $S$.

Definition 7.2.5. Let $M''_n$ be the union of the irreducible components of $M'_n$ which dominate $S$. This union is nonempty because $M'_n$ dominates $S$.

Proposition 7.2.6. Let $\{x'_n \to M''_n\}_n$ be a coherent sequence of points valued in finite extensions $k'_n$ of $k$, and put $x' = \lim_{\longleftarrow} x'_n$. Let $x \in S$ be the image of $x'$. Let $\mathcal{F}$ be the object of $F^\dagger\text{-Isoc}(U \times_S x') \otimes L$ arising from this sequence via Proposition 6.7.4. Then $\mathcal{F}$ is the base extension of an object of $F^\dagger\text{-Isoc}(U \times_S x) \otimes L$ which is a companion of $\mathcal{E}|_{U \times_S x'}$. 

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Proof. Suppose first that \( x \in W \). For any \( m \geq n \), the morphism \( M''/m \to M''/n \) is surjective. Consequently, given a point \( x'/m \to M''/n \) valued in a finite extension \( k'/m \) of \( k \), we can lift it to a tower of points \( \{ x'_m \to M''/n \}_{m \geq n} \), each valued in a finite extension \( k'_m \) of \( k'_n \). Let \( k' \) be the union of the \( k'_m \) and put \( x' = \text{Spec}(k') \); taking the limit over the \( x'_m \) and applying Proposition 6.7.4, we obtain an object \( F \) of \( \mathbf{F-Isoc}((U \times_S x'/m) \otimes L) \). By Lemma 7.2.3, Proposition 3.2.3, and the Chebotarev density theorem, the first steps of the slope filtration of \( F \) and the companion \( F' \) (which both have slope 0 because \( x'_n \) lies over \( W^{(n)} \)) have isomorphic semisimplifications in \( \mathbf{F-Isoc}(W) \otimes L \). By Theorem 3.3.1, \( F \) and \( F' \) are isomorphic in \( \mathbf{F-Isoc}((U \times_S x'/m) \otimes \overline{\mathbb{Q}}_p) \), proving the claim in this case.

To prove the general case, it will suffice to check that for any curve \( C \in S \), if there is a finite subset \( T \) of \( C \) such that the claim holds whenever \( x \in C \setminus T \), then the claim holds whenever \( x \in C \). To this end, note that for \( x \in C \setminus T \), every point of \( M''/n \) lying along \( x \) has residue field \( x \), so (by Chebotarev density again) every component of \( (M''/n \times_C C)_{\text{red}} \) is actually isomorphic to \( C \). Consequently, by taking the inverse limit, we obtain an object of \( \mathbf{F-Isoc}((U \times_S C) \otimes L) \) whose restriction to \( \mathbf{F-Isoc}((U \times_S (C \setminus T)) \otimes L \) is a companion of \( E(S \setminus T) \). By Lemma 2.3.3, the companion relation extends over \( U \times S \), proving the claim. \( \square \)

Proposition 7.2.7. For each irreducible component \( Y \) of \( M''/n \), the projection \( Y \to S \) induces an isomorphism \( Y_{\text{red}} \to S \).

Proof. By Proposition 7.2.6, every point of \( M''/n \) valued in a finite extension of \( k \) has the same residue field as its image in \( S \). This implies firstly that \( M''/n \) is finite over \( S \), and secondly that each component of \( M''/n \) becomes isomorphic to \( S \) at the level of reduced substacks. \( \square \)

Remark 7.2.8. Reprising the proof of Proposition 7.2.6, if we now choose a coherent sequence of irreducible components of \( M''/n \), by Proposition 7.2.7 we may take the inverse limit over the reduced substacks to obtain an object of \( \mathbf{F-Isoc}((U) \otimes L) \) which is a crystalline companion of \( E \).

8. Companions and corollaries

With the key result in hand, we complete the construction of crystalline companions, and record some old and new corollaries.

8.1. Proof of the main theorems. We complete the proofs of Theorem 0.1.1 and Theorem 0.1.2.

Theorem 8.1.1. Any algebraic étale coefficient object on \( X \) admits a crystalline companion.

Proof. We may assume at once that \( X \) is connected; by Remark 2.1.7, we may also assume that \( X \) is geometrically irreducible. Let \( E \) be an algebraic étale coefficient object on \( X \); we may assume that \( E \) is absolutely irreducible. By a constant twist, we may ensure that the least generic slope of \( E \) (Lemma-Definition 2.6.1) is 0.

By Lemma 2.3.5, we may check the claim after replacing \( X \) with an alteration or an open dense subspace. By Proposition 1.3.2 and Lemma 2.4.2, after replacing \( X \) by an alteration, we may ensure that \( E \) is docile. By Corollary 1.3.4, we may assume that \( X \) admits a good compactification \( (\bar{X}, \bar{Z}) \) which form a stable \( n \)-pointed genus-\( g \) fibration over some base \( S \). In light of Remark 1.3.5, we may also assume that \( X \to S \) admits a section on which \( E \) is constant. By Lemma 2.2.4, the fibers of \( E \) over \( S \) are also absolutely irreducible.
We are now in the situation of Hypothesis 7.1.1. As in Remark 7.2.8, Proposition 7.2.7 implies the existence of a crystalline companion of $E$. □

**Corollary 8.1.2.** Theorem 0.1.2 holds in the case $\ell = p$.

*Proof.* For $\ell \neq p$, this is included in Theorem 8.1.1. For $\ell = p$, we may first apply Corollary 2.5.2 to change to the case $\ell \neq p$, then proceed as before. □

We mention explicitly the following special case of Theorem 0.1.2.

**Corollary 8.1.3.** Let $E$ be an $E$-algebraic coefficient object on $X$. Then for any automorphism $\tau$ of $E$, there exists a coefficient object $E_{\tau}$ such that for each $x \in X^{\circ}$, we have the equality $P((E_{\tau})_x, T) = \tau(P(E_x, T))$ in $E[T]$ (where $\tau$ acts coefficientwise).

**Corollary 8.1.4.** Theorem 0.1.1 holds in all cases.

*Proof.* By Corollary 2.5.2, parts (i)–(v) hold. To prove (vi), note that (ii) implies that $E$ is algebraic, so Corollary 8.1.2 implies the existence of a crystalline companion $F$ (more precisely, if $\ell = p$ we must apply Theorem 0.1.1(v) to switch to an étale companion first). It remains to check that $F$ is irreducible and that $\det(F)$ is of finite order.

If $F$ were reducible, we could apply (v) to it to get a reducible companion $E'$ in the same category as $E$; by the uniqueness of semisimple companions (Lemma 2.3.2(c)), this would imply that $E$ is reducible. We thus deduce that $F$ is irreducible. (Compare [37, Lemma 3.6.1].)

To check that $\det(F)$ is of finite order, note that it is a companion of $\det(E)$. By the same token, if we choose a positive integer $n$ such that $\det(E)^{\otimes n}$ is trivial, then it has $\det(F)^{\otimes n}$ as a companion; the latter must then be trivial again by the uniqueness of semisimple companions. (Compare [37, Corollary 3.2.7].) □

8.2. **Newton polygons revisited.** With Theorem 0.1.2 in hand, we can now assert much stronger properties of Newton polygons of Weil sheaves than were asserted in Lemma-Definition 2.6.1. These extend the Grothendieck-Katz semicontinuity theorem and the de Jong–Oort–Yang purity theorem to étale coefficients.

**Theorem 8.2.1.** Let $E$ be an $E$-algebraic $\ell$-adic coefficient object for some number field $E$, and fix an embedding $E \hookrightarrow \overline{\mathbb{Q}}_p$. Then the function $x \mapsto N_x(E)$ from Lemma-Definition 2.6.1 on $X$ is upper semicontinuous, with the endpoints being locally constant; in particular, this function defines a locally closed stratification of $X$.

*Proof.* By Theorem 0.1.2, this follows from Lemma 3.1.2. □

**Theorem 8.2.2.** With notation as in Theorem 8.2.1, the Newton polygon stratification jumps purely in codimension 1. More precisely, for $X$ irreducible, each breakpoint of the generic Newton polygon disappears purely in codimension 1.

*Proof.* By Theorem 0.1.2, this follows from Lemma 3.1.2. □

**Remark 8.2.3.** Suppose that $X$ admits a good compactification $\overline{X}$ and that $E$ is a docile coefficient object on $X$. Apply Theorem 0.1.2 to construct a crystalline companion $F$ of $E$; by Corollary 2.5.4, $F$ is again docile. We then define the Newton polygon function $N_x(E) := N_x(F)$; by retracing through the arguments cited in [36, §3], it can be shown that it satisfies the analogues of the theorems of Grothendieck-Katz and de Jong–Oort–Yang. In particular, the conclusions of Theorem 8.2.1 and Theorem 8.2.2 can be seen to
carry over to this definition; we leave a detailed development of this statement to a later occasion.

Example 8.2.4. Take $X = \mathbb{P}^1_k \setminus \{0, 1, \infty\}$ with coordinate $\lambda$ and let $\mathcal{E}$ be the middle cohomology of the Legendre elliptic curve $y^2 = x(x - 1)(x - \lambda)$. (This is similar to [36, Example 4.6] except with $N = 2$.) Using the Tate uniformization of elliptic curves with split multiplicative reduction, one can show that $\mathcal{E}$ is docile and for $\lambda \in \{0, 1, \infty\}$, $N_\lambda(\mathcal{E})$ equals the generic value (i.e., its slopes are 0 and 1).

8.3. Wan’s theorem on fixed-slope $L$-functions. As observed in [37], another result that can be transferred from crystalline to étale coefficients using Theorem 0.1.2 is Wan’s theorem (formerly Dwork’s conjecture) on fixed-slope $L$-functions.

Theorem 8.3.1. Let $\mathcal{E}$ be an algebraic coefficient object on $X$. For $s \in \mathbb{Q}$, let $P_s(\mathcal{E}_x, T)$ be the factor of $P(\mathcal{E}_x, T)$ with constant term 1 corresponding to the slope-$s$ segment of $N_x(\mathcal{E})$. Then the associated $L$-function

$$L_s(X, \mathcal{E}, T) = \prod_{x \in X^0} P_s(\mathcal{E}_x, T^{\kappa(x):k})^{-1}$$

is $p$-adic meromorphic (i.e., a ratio of two $p$-adic entire series).

Proof. For any locally closed stratification of $X$, $L_s(X, \mathcal{E}, T)$ equals the product of $L_s(Y, \mathcal{E}, T)$ as $Y$ varies over the strata; we may thus assume that $X$ is affine. Apply Theorem 0.1.2 to construct a crystalline companion $\mathcal{F}$ of $\mathcal{E}$; we then have $L_s(X, \mathcal{E}, T) = L_s(X, \mathcal{F}, T)$. The $p$-adic meromorphicity of $L_s(X, \mathcal{F}, T)$ is a theorem of Wan [64, Theorem 1.1] (see also [63, 65]); this proves the claim. □

References


