The 52nd William Lowell Putnam Mathematical Competition
Saturday, December 7, 1991

A–1 A 2×3 rectangle has vertices as $(0,0), (2,0), (0,3)$, and $(2,3)$. It rotates 90° clockwise about the point $(2,0)$. It then rotates 90° clockwise about the point $(5,0)$, then 90° clockwise about the point $(7,0)$, and finally, 90° clockwise about the point $(10,0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is (1,1).

A–2 Let $\mathbf{A}$ and $\mathbf{B}$ be different $n \times n$ matrices with real entries. If $\mathbf{A}^3 = \mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B} = \mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2 + \mathbf{B}^2$ be invertible?

A–3 Find all real polynomials $p(x)$ of degree $n \geq 2$ for which there exist real numbers $r_1 < r_2 < \cdots < r_n$ such that

1. $p(r_i) = 0, \qquad i = 1, 2, \dots, n$, and
2. $p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0 \qquad i = 1, 2, \dots, n-1,$

where $p'(x)$ denotes the derivative of $p(x)$.

A–4 Does there exist an infinite sequence of closed discs $D_1, D_2, D_3, \dots$ in the plane, with centers $c_1, c_2, c_3, \dots$, respectively, such that

1. the c_i have no limit point in the finite plane,
2. the sum of the areas of the D_i is finite, and
3. every line in the plane intersects at least one of the D_i?

A–5 Find the maximum value of

$$\int_0^y \sqrt{x^4 + (y - y^2)^2}\, dx$$

for $0 \leq y \leq 1$.

A–6 Let $A(n)$ denote the number of sums of positive integers

$$a_1 + a_2 + \cdots + a_r$$

which add up to n with

$$a_1 > a_2 + a_3, a_2 > a_3 + a_4, \dots,$$
$$a_{r-2} > a_{r-1} + a_r, a_{r-1} > a_r.$$

Let $B(n)$ denote the number of $b_1 + b_2 + \cdots + b_s$ which add up to n, with

1. $b_1 \geq b_2 \geq \cdots \geq b_s$,
2. each b_i is in the sequence $1, 2, 4, \dots, g_j, \dots$ defined by $g_1 = 1$, $g_2 = 2$, and $g_j = g_{j-1} + g_{j-2} + 1$, and
3. if $b_1 = g_k$ then every element in $\{1, 2, 4, \dots, g_k\}$ appears at least once as a b_i.

Prove that $A(n) = B(n)$ for each $n \geq 1$.

(For example, $A(7) = 5$ because the relevant sums are $7, 6+1, 5+2, 4+3, 4+2+1$, and $B(7) = 5$ because the relevant sums are $4+2+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1$.)

B–1 For each integer $n \geq 0$, let $S(n) = n - m^2$, where m is the greatest integer with $m^2 \leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0 = A$ and $a_{k+1} = a_k + S(a_k)$ for $k \geq 0$. For what positive integers A is this sequence eventually constant?

B–2 Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers x and y,

$$f(x+y) = f(x)f(y) - g(x)g(y),$$
$$g(x+y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all x.

B–3 Does there exist a real number L such that, if m and n are integers greater than L, then an $m \times n$ rectangle may be expressed as a union of 4×6 and 5×7 rectangles, any two of which intersect at most along their boundaries?

B–4 Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p} \binom{p}{j} \binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

B–5 Let p be an odd prime and let $\mathbb{Z}_p$ denote (the field of) integers modulo p. How many elements are in the set

$$\{x^2 : x \in \mathbb{Z}_p\} \cap \{y^2 + 1 : y \in \mathbb{Z}_p\}?$$

B–6 Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that

$$a^x b^{1-x} \leq a \frac{\sinh ux}{\sinh u} + b \frac{\sinh u(1-x)}{\sinh u}$$

for all u with $0 < |u| \leq c$ and for all x, $0 < x < 1$. (Note: $\sinh u = (e^u - e^{-u})/2$.)