Solutions to the 57th William Lowell Putnam Mathematical Competition
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A–1 If x and y are the sides of two squares with combined area 1, then $x^2 + y^2 = 1$. Suppose without loss of generality that $x \geq y$. Then the shorter side of a rectangle containing both squares without overlap must be at least x, and the longer side must be at least $x + y$. Hence the desired value of A is the maximum of $x(x+y)$.

To find this maximum, we let $x = \cos\theta, y = \sin\theta$ with $\theta \in [0, \pi/4]$. Then we are to maximize

$$\cos^2\theta + \sin\theta\cos\theta = \frac{1}{2}(1 + \cos 2\theta + \sin 2\theta)$$
$$= \frac{1}{2} + \frac{\sqrt{2}}{2}\cos(2\theta - \pi/4)$$
$$\leq \frac{1 + \sqrt{2}}{2},$$

with equality for $\theta = \pi/8$. Hence this value is the desired value of A.

A–2 Let O_1 and O_2 be the centers of C_1 and C_2, respectively. (We are assuming C_1 has radius 1 and C_2 has radius 3.) Then the desired locus is an annulus centered at the midpoint of O_1O_2, with inner radius 1 and outer radius 2.

For a fixed point Q on C_2, the locus of the midpoints of the segments PQ for P lying on C_1 is the image of C_1 under a homothety centered at Q of radius $1/2$, which is a circle of radius $1/2$. As Q varies, the center of this smaller circle traces out a circle C_3 of radius $3/2$ (again by homothety). By considering the two positions of Q on the line of centers of the circles, one sees that C_3 is centered at the midpoint of O_1O_2, and the locus is now clearly the specified annulus.

A–3 The claim is false. There are $\binom{6}{3} = 20$ ways to choose 3 of the 6 courses; have each student choose a different set of 3 courses. Then each pair of courses is chosen by 4 students (corresponding to the four ways to complete this pair to a set of 3 courses) and is not chosen by 4 students (corresponding to the 3-element subsets of the remaining 4 courses).

Note: Assuming that no two students choose the same courses, the above counterexample is unique (up to permuting students). This may be seen as follows: Given a group of students, suppose that for any pair of courses (among the six) there are at most 4 students taking both, and at most 4 taking neither. Then there are at most $120 = (4+4)\binom{6}{2}$ pairs (s, p), where s is a student, and p is a set of two courses of which s is taking either both or none. On the other hand, if a student s is taking k courses, then he/she occurs in $f(k) = \binom{k}{2} + \binom{6-k}{2}$ such

pairs (s, p). As $f(k)$ is minimized for $k = 3$, it follows that every student occurs in at least $6 = \binom{3}{2} + \binom{3}{2}$ such pairs (s, p). Hence there can be at most $120/6 = 20$ students, with equality only if each student takes 3 courses, and for each set of two courses, there are exactly 4 students who take both and exactly 4 who take neither. Since there are only 4 ways to complete a given pair of courses to a set of 3, and only 4 ways to choose 3 courses not containing the given pair, the only way for there to be 20 students (under our hypotheses) is if all sets of 3 courses are in fact taken. This is the desired conclusion.

However, Robin Chapman has pointed out that the solution is not unique in the problem as stated, because a given selection of courses may be made by more than one student. One alternate solution is to identify the 6 courses with pairs of antipodal vertices of an icosahedron, and have each student pick a different face and choose the three vertices touching that face. In this example, each of 10 selections is made by a pair of students.

A–4 In fact, we will show that such a function g exists with the property that $(a, b, c) \in S$ if and only if $g(d) < g(e) < g(f)$ for some cyclic permutation (d, e, f) of (a, b, c). We proceed by induction on the number of elements in A. If $A = \{a, b, c\}$ and $(a, b, c) \in S$, then choose g with $g(a) < g(b) < g(c)$, otherwise choose g with $g(a) > g(b) > g(c)$.

Now let z be an element of A and $B = A - \{z\}$. Let $a_1, \ldots, a_n$ be the elements of B labeled such that $g(a_1) < g(a_2) < \cdots < g(a_n)$. We claim that there exists a unique $i \in \{1, \ldots, n\}$ such that $(a_i, z, a_{i+1}) \in S$, where hereafter $a_{n+k} = a_k$.

We show existence first. Suppose no such i exists; then for all $i, k \in \{1, \ldots, n\}$, we have $(a_{i+k}, z, a_i) \notin S$. This holds by property 1 for $k = 1$ and by induction on k in general, noting that

$$(a_{i+k+1}, z, a_{i+k}), (a_{i+k}, z, a_i) \in S$$
$$\Rightarrow (a_{i+k}, a_{i+k+1}, z), (z, a_i, a_{i+k}) \in S$$
$$\Rightarrow (a_{i+k+1}, z, a_i) \in S.$$

Applying this when $k = n$, we get $(a_{i-1}, z, a_i) \in S$, contradicting the fact that $(a_i, z, a_{i-1}) \in S$. Hence existence follows.

Now we show uniqueness. Suppose $(a_i, z, a_{i+1}) \in S$; then for any $j \neq i - 1, i, i + 1$, we have $(a_i, a_{i+1}, a_j), (a_j, a_{j+1}, a_i) \in S$ by the assumption on G.

Therefore

$$(a_i, z, a_{i+1}), (a_{i+1}, a_j, a_i) \in S \Rightarrow (a_j, a_i, z) \in S$$
$$(a_i, z, a_j), (a_j, a_{j+1}, a_i) \in S \Rightarrow (z, a_j, a_{j+1}),$$

so $(a_j, z, a_{j+1}) \notin S$. The case $j = i+1$ is ruled out by

$$(a_i, z, a_{i+1}), (a_{i+1}, a_{i+2}, a_i) \in S \Rightarrow (z, a_{i+1}, a_{i+2}) \in S$$

and the case $j = i-1$ is similar.

Finally, we put $g(z)$ in $(g(a_n), +\infty)$ if $i = n$, and $(g(a_i), g(a_{i+1}))$ otherwise; an analysis similar to that above shows that g has the desired property.

A–5 (due to Lenny Ng) For $1 \le n \le p-1$, p divides $\binom{p}{n}$ and

$$\frac{1}{p}\binom{p}{n} = \frac{1}{n}\frac{p-1}{1}\frac{p-2}{2}\cdots\frac{p-n+1}{n-1}$$
$$\equiv \frac{(-1)^{n-1}}{n} \pmod{p},$$

where the congruence $x \equiv y \pmod{p}$ means that $x - y$ is a rational number whose numerator, in reduced form, is divisible by p. Hence it suffices to show that

$$\sum_{n=1}^{k} \frac{(-1)^{n-1}}{n} \equiv 0 \pmod{p}.$$

We distinguish two cases based on $p \pmod 6$. First suppose $p = 6r+1$, so that $k = 4r$. Then

$$\sum_{n=1}^{4r} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{4r} \frac{1}{n} - 2\sum_{n=1}^{2r} \frac{1}{2n}$$
$$= \sum_{n=1}^{2r} \left(\frac{1}{n} - \frac{1}{n}\right) + \sum_{n=2r+1}^{3r} \left(\frac{1}{n} + \frac{1}{6r+1-n}\right)$$
$$= \sum_{n=2r+1}^{3r} \frac{p}{n(p-n)} \equiv 0 \pmod{p},$$

since $p = 6r+1$.

Now suppose $p = 6r+5$, so that $k = 4r+3$. A similar argument gives

$$\sum_{n=1}^{4r+3} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{4r+3} \frac{1}{n} + 2\sum_{n=1}^{2r+1} \frac{1}{2n}$$
$$= \sum_{n=1}^{2r+1} \left(\frac{1}{n} - \frac{1}{n}\right) + \sum_{n=2r+2}^{3r+2} \left(\frac{1}{n} + \frac{1}{6r+5-n}\right)$$
$$= \sum_{n=2r+2}^{3r+2} \frac{p}{n(p-n)} \equiv 0 \pmod{p}.$$

A–6 We first consider the case $c \le 1/4$; we shall show in this case f must be constant. The relation

$$f(x) = f(x^2 + c) = f((-x)^2 + c) = f(-x)$$

proves that f is an even function. Let $r_1 \le r_2$ be the roots of $x^2 + c - x$, both of which are real. If $x > r_2$,

define $x_0 = x$ and $x_{n+1} = \sqrt{x_n - c}$ for each positive integer x. By induction on n, $r_2 < x_{n+1} < x_n$ for all n, so the sequence $\{x_n\}$ tends to a limit L which is a root of $x^2 + c = x$ not less than r_2. Of course this means $L = r_2$. Since $f(x) = f(x_n)$ for all n and $x_n \to r_2$, we conclude $f(x) = f(r_2)$, so f is constant on $x \ge r_2$.

If $r_1 < x < r_2$ and x_n is defined as before, then by induction, $x_n < x_{n+1} < r_2$. Note that the sequence can be defined because $r_1 > c$; the latter follows by noting that the polynomial $x^2 - x + c$ is positive at $x = c$ and has its minimum at $1/2 > c$, so both roots are greater than c. In any case, we deduce that $f(x)$ is also constant on $r_1 \le x \le r_2$.

Finally, suppose $x < r_1$. Now define $x_0 = x, x_{n+1} = x_n^2 + c$. Given that $x_n < r_1$, we have $x_{n+1} > x_n$. Thus if we had $x_n < r_1$ for all n, by the same argument as in the first case we deduce $x_n \to r_1$ and so $f(x) = f(r_1)$. Actually, this doesn't happen; eventually we have $x_n > r_1$, in which case $f(x) = f(x_n) = f(r_1)$ by what we have already shown. We conclude that f is a constant function. (Thanks to Marshall Buck for catching an inaccuracy in a previous version of this solution.)

Now suppose $c > 1/4$. Then the sequence x_n defined by $x_0 = 0$ and $x_{n+1} = x_n^2 + c$ is strictly increasing and has no limit point. Thus if we define f on $[x_0, x_1]$ as any continuous function with equal values on the endpoints, and extend the definition from $[x_n, x_{n+1}]$ to $[x_{n+1}, x_{n+2}]$ by the relation $f(x) = f(x^2 + c)$, and extend the definition further to $x < 0$ by the relation $f(x) = f(-x)$, the resulting function has the desired property. Moreover, any function with that property clearly has this form.

B–1 Let $[n]$ denote the set $\{1, 2, \ldots, n\}$, and let f_n denote the number of minimal selfish subsets of $[n]$. Then the number of minimal selfish subsets of $[n]$ not containing n is equal to f_{n-1}. On the other hand, for any minimal selfish subset of $[n]$ containing n, by subtracting 1 from each element, and then taking away the element $n-1$ from the set, we obtain a minimal selfish subset of $[n-2]$ (since 1 and n cannot both occur in a selfish set). Conversely, any minimal selfish subset of $[n-2]$ gives rise to a minimal selfish subset of $[n]$ containing n by the inverse procedure. Hence the number of minimal selfish subsets of $[n]$ containing n is f_{n-2}. Thus we obtain $f_n = f_{n-1} + f_{n-2}$. Since $f_1 = f_2 = 1$, we have $f_n = F_n$, where F_n denotes the nth term of the Fibonacci sequence.

B–2 By estimating the area under the graph of $\ln x$ using upper and lower rectangles of width 2, we get

$$\int_1^{2n-1} \ln x \, dx \le 2(\ln(3) + \cdots + \ln(2n-1))$$
$$\le \int_3^{2n+1} \ln x \, dx.$$

Since $\int \ln x \, dx = x \ln x - x + C$, we have, upon exponen-

tiating and taking square roots,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < (2n-1)^{\frac{2n-1}{2}} e^{-n+1}$$

$$\leq 1 \cdot 3 \cdots (2n-1)$$

$$\leq (2n+1)^{\frac{2n+1}{2}} \frac{e^{-n+1}}{3^{3/2}}$$

$$< \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}},$$

using the fact that $1 < e < 3$.

B–3 View $x_1, \ldots, x_n$ as an arrangement of the numbers $1, 2, \ldots, n$ on a circle. We prove that the optimal arrangement is

$$\ldots, n-4, n-2, n, n-1, n-3, \ldots$$

To show this, note that if a, b is a pair of adjacent numbers and c, d is another pair (read in the same order around the circle) with $a < d$ and $b > c$, then the segment from b to c can be reversed, increasing the sum by

$$ac + bd - ab - cd = (d-a)(b-c) > 0.$$

Now relabel the numbers so they appear in order as follows:

$$\ldots, a_{n-4}, a_{n-2}, a_n = n, a_{n-1}, a_{n-3}, \ldots$$

where without loss of generality we assume $a_{n-1} > a_{n-2}$. By considering the pairs a_{n-2}, a_n and a_{n-1}, a_{n-3} and using the trivial fact $a_n > a_{n-1}$, we deduce $a_{n-2} > a_{n-3}$. We then compare the pairs a_{n-4}, a_{n-2} and a_{n-1}, a_{n-3}, and using that $a_{n-1} > a_{n-2}$, we deduce $a_{n-3} > a_{n-4}$. Continuing in this fashion, we prove that $a_n > a_{n-1} > \cdots > a_1$ and so $a_k = k$ for $k = 1, 2, \ldots, n$, i.e. that the optimal arrangement is as claimed. In particular, the maximum value of the sum is

$$1 \cdot 2 + (n-1) \cdot n + 1 \cdot 3 + 2 \cdot 4 + \cdots + (n-2) \cdot n$$

$$= 2 + n^2 - n + (1^2 - 1) + \cdots + [(n-1)^2 - 1]$$

$$= n^2 - n + 2 - (n-1) + \frac{(n-1)n(2n-1)}{6}$$

$$= \frac{2n^3 + 3n^2 - 11n + 18}{6}.$$

Alternate solution: We prove by induction that the value given above is an upper bound; it is clearly a lower bound because of the arrangement given above. Assume this is the case for $n-1$. The optimal arrangement for n is obtained from some arrangement for $n-1$ by inserting n between some pair x, y of adjacent terms. This operation increases the sum by $nx + ny - xy = n^2 - (n-x)(n-y)$, which is an increasing function of both x and y. In particular, this difference is maximal

when x and y equal $n-1$ and $n-2$. Fortunately, this yields precisely the difference between the claimed upper bound for n and the assumed upper bound for $n-1$, completing the induction.

B–4 Suppose such a matrix A exists. If the eigenvalues of A (over the complex numbers) are distinct, then there exists a complex matrix C such that $B = CAC^{-1}$ is diagonal. Consequently, $\sin B$ is diagonal. But then $\sin A = C^{-1}(\sin B)C$ must be diagonalizable, a contradiction. Hence the eigenvalues of A are the same, and A has a conjugate $B = CAC^{-1}$ over the complex numbers of the form

$$\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}.$$

A direct computation shows that

$$\sin B = \begin{pmatrix} \sin x & y \cdot \cos x \\ 0 & \sin x \end{pmatrix}.$$

Since $\sin A$ and $\sin B$ are conjugate, their eigenvalues must be the same, and so we must have $\sin x = 1$. This implies $\cos x = 0$, so that $\sin B$ is the identity matrix, as must be $\sin A$, a contradiction. Thus A cannot exist.

Alternate solution (due to Craig Helfgott and Alex Popa): Define both $\sin A$ and $\cos A$ by the usual power series. Since A commutes with itself, the power series identity

$$\sin^2 A + \cos^2 A = I$$

holds. But if $\sin A$ is the given matrix, then by the above identity, $\cos^2 A$ must equal $\begin{pmatrix} 0 & -2 \cdot 1996 \\ 0 & 0 \end{pmatrix}$ which is a nilpotent matrix. Thus $\cos A$ is also nilpotent. However, the square of any 2×2 nilpotent matrix must be zero (e.g., by the Cayley-Hamilton theorem). This is a contradiction.

B–5 Consider a $1 \times n$ checkerboard, in which we write an n-letter string, one letter per square. If the string is balanced, we can cover each pair of adjacent squares containing the same letter with a 1×2 domino, and these will not overlap (because no three in a row can be the same). Moreover, any domino is separated from the next by an even number of squares, since they must cover opposite letters, and the sequence must alternate in between.

Conversely, any arrangement of dominoes where adjacent dominoes are separated by an even number of squares corresponds to a unique balanced string, once we choose whether the string starts with X or O. In other words, the number of balanced strings is twice the number of acceptable domino arrangements.

We count these arrangements by numbering the squares $0, 1, \ldots, n-1$ and distinguishing whether the dominoes start on even or odd numbers. Once this is decided, one

simply chooses whether or not to put a domino in each eligible position. Thus we have $2^{\lfloor n/2 \rfloor}$ arrangements in the first case and $2^{\lfloor (n-1)/2 \rfloor}$ in the second, but note that the case of no dominoes has been counted twice. Hence the number of balanced strings is

$$2^{\lfloor (n+2)/2 \rfloor} + 2^{\lfloor (n+1)/2 \rfloor} - 2.$$

B–6 We will prove the claim assuming only that the convex hull of the points (a_i, b_i) contains the origin in its interior. (Thanks to Marshall Buck for pointing out that the last three words are necessary in the previous sentence!) Let $u = \log x, v = \log y$ so that the left-hand side of the given equation is

$$(a_1, b_1) \exp(a_1 u + b_1 v) + (a_2, b_2) \exp(a_2 u + b_2 v) +$$
$$\cdots + (a_n, b_n) \exp(a_n u + b_n v). \quad (1)$$

Now note that (1) is the gradient of the function

$$f(u, v) = exp(a_1 u + b_1 v) + exp(a_2 u + b_2 v) +$$
$$\cdots + exp(a_n u + b_n v),$$

and so it suffices to show f has a critical point. We will in fact show f has a global minimum.

Clearly we have

$$f(u, v) \geq \exp\left(\max_i (a_i u + b_i v)\right).$$

Note that this maximum is positive for $(u, v) \neq (0, 0)$: if we had $a_i u + b_i v < 0$ for all i, then the subset $ur + vs < 0$ of the rs-plane would be a half-plane containing all of the points (a_i, b_i), whose convex hull would then not contain the origin, a contradiction.

The function $\max_i (a_i u + b_i v)$ is clearly continuous on the unit circle $u^2 + v^2 = 1$, which is compact. Hence it has a global minimum $M > 0$, and so for all u, v,

$$\max_i (a_i u + b_i v) \geq M \sqrt{u^2 + v^2}.$$

In particular, $f \geq n + 1$ on the disk of radius $\sqrt{(n+1)/M}$. Since $f(0, 0) = n$, the infimum of f is the same over the entire uv-plane as over this disk, which again is compact. Hence f attains its infimal value at some point in the disk, which is the desired global minimum.

Noam Elkies has suggested an alternate solution as follows: for $r > 0$, draw the loop traced by (1) as (u, v) travels counterclockwise around the circle $u^2 + v^2 = r^2$. For $r = 0$, this of course has winding number 0 about any point, but for r large, one can show this loop has winding number 1 about the origin, so somewhere in between the loop must pass through the origin. (Proving this latter fact is a little tricky.)