The 59th William Lowell Putnam Mathematical Competition
Saturday, December 5, 1998

A–1 A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

A–2 Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A+B$ depends only on the arc length, and not on the position, of s.

A–3 Let f be a real function on the real line with continuous third derivative. Prove that there exists a point a such that

$$f(a)\cdot f'(a)\cdot f''(a)\cdot f'''(a)\geq 0.$$

A–4 Let $A_1=0$ and $A_2=1$. For $n>2$, the number A_n is defined by concatenating the decimal expansions of A_{n-1} and A_{n-2} from left to right. For example $A_3=A_2A_1=10$, $A_4=A_3A_2=101$, $A_5=A_4A_3=10110$, and so forth. Determine all n such that 11 divides A_n.

A–5 Let $\mathscr{F}$ be a finite collection of open discs in $\mathbb{R}^2$ whose union contains a set $E\subseteq\mathbb{R}^2$. Show that there is a pairwise disjoint subcollection $D_1,\ldots,D_n$ in $\mathscr{F}$ such that

$$E\subseteq\cup_{j=1}^{n}3D_j.$$

Here, if D is the disc of radius r and center P, then $3D$ is the disc of radius $3r$ and center P.

A–6 Let A,B,C denote distinct points with integer coordinates in $\mathbb{R}^2$. Prove that if

$$(|AB|+|BC|)^2<8\cdot[ABC]+1$$

then A,B,C are three vertices of a square. Here $|XY|$ is the length of segment XY and $[ABC]$ is the area of triangle ABC.

B–1 Find the minimum value of

$$\frac{(x+1/x)^6-(x^6+1/x^6)-2}{(x+1/x)^3+(x^3+1/x^3)}$$

for $x>0$.

B–2 Given a point (a,b) with $0<b<a$, determine the minimum perimeter of a triangle with one vertex at (a,b), one on the x-axis, and one on the line $y=x$. You may assume that a triangle of minimum perimeter exists.

B–3 let H be the unit hemisphere $\{(x,y,z):x^2+y^2+z^2=1,z\geq 0\}$, C the unit circle $\{(x,y,0):x^2+y^2=1\}$, and P the regular pentagon inscribed in C. Determine the surface area of that portion of H lying over the planar region inside P, and write your answer in the form $A\sin\alpha+B\cos\beta$, where A,B,α,β are real numbers.

B–4 Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1}(-1)^{\lfloor i/m\rfloor+\lfloor i/n\rfloor}=0.$$

B–5 Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N=1111\cdots 11.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

B–6 Prove that, for any integers a,b,c, there exists a positive integer n such that $\sqrt{n^3+an^2+bn+c}$ is not an integer.