The 60th William Lowell Putnam Mathematical Competition
Saturday, December 4, 1999

A–1 Find polynomials $f(x)$,$g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)|-|g(x)|+h(x)=\begin{cases}-1 & \text{if } x<-1\\ 3x+2 & \text{if } -1\le x\le 0\\ -2x+2 & \text{if } x>0.\end{cases}$$

A–2 Let $p(x)$ be a polynomial that is nonnegative for all real x. Prove that for some k, there are polynomials $f_1(x),\dots,f_k(x)$ such that

$$p(x)=\sum_{j=1}^{k}(f_j(x))^2.$$

A–3 Consider the power series expansion

$$\frac{1}{1-2x-x^2}=\sum_{n=0}^{\infty}a_nx^n.$$

Prove that, for each integer $n\ge 0$, there is an integer m such that

$$a_n^2+a_{n+1}^2=a_m.$$

A–4 Sum the series

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n}{3^m(n3^m+m3^n)}.$$

A–5 Prove that there is a constant C such that, if $p(x)$ is a polynomial of degree 1999, then

$$|p(0)|\le C\int_{-1}^{1}|p(x)|\,dx.$$

A–6 The sequence $(a_n)_{n\ge 1}$ is defined by $a_1=1, a_2=2, a_3=24$, and, for $n\ge 4$,

$$a_n=\frac{6a_{n-1}^2a_{n-3}-8a_{n-1}a_{n-2}^2}{a_{n-2}a_{n-3}}.$$

Show that, for all n, a_n is an integer multiple of n.

B–1 Right triangle ABC has right angle at C and $\angle BAC=\theta$; the point D is chosen on AB so that $|AC|=|AD|=1$; the point E is chosen on BC so that $\angle CDE=\theta$. The perpendicular to BC at E meets AB at F. Evaluate $\lim_{\theta\to 0}|EF|$.

B–2 Let $P(x)$ be a polynomial of degree n such that $P(x)=Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

B–3 Let $A=\{(x,y):0\le x,y<1\}$. For $(x,y)\in A$, let

$$S(x,y)=\sum_{\frac{1}{2}\le\frac{m}{n}\le 2}x^my^n,$$

where the sum ranges over all pairs (m,n) of positive integers satisfying the indicated inequalities. Evaluate

$$\lim_{(x,y)\to(1,1),(x,y)\in A}(1-xy^2)(1-x^2y)S(x,y).$$

B–4 Let f be a real function with a continuous third derivative such that $f(x),f'(x),f''(x),f'''(x)$ are positive for all x. Suppose that $f'''(x)\le f(x)$ for all x. Show that $f'(x)<2f(x)$ for all x.

B–5 For an integer $n\ge 3$, let $\theta=2\pi/n$. Evaluate the determinant of the $n\times n$ matrix $I+A$, where I is the $n\times n$ identity matrix and $A=(a_{jk})$ has entries $a_{jk}=\cos(j\theta+k\theta)$ for all j,k.

B–6 Let S be a finite set of integers, each greater than 1. Suppose that for each integer n there is some $s\in S$ such that $\gcd(s,n)=1$ or $\gcd(s,n)=s$. Show that there exist $s,t\in S$ such that $\gcd(s,t)$ is prime.