The 61st William Lowell Putnam Mathematical Competition
Saturday, December 2, 2000

A–1 Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \dots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

A–2 Prove that there exist infinitely many integers n such that $n, n+1, n+2$ are each the sum of the squares of two integers. [Example: $0 = 0^2 + 0^2$, $1 = 0^2 + 1^2$, $2 = 1^2 + 1^2$.]

A–3 The octagon $P_1P_2P_3P_4P_5P_6P_7P_8$ is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon $P_1P_3P_5P_7$ is a square of area 5, and the polygon $P_2P_4P_6P_8$ is a rectangle of area 4, find the maximum possible area of the octagon.

A–4 Show that the improper integral

$$\lim_{B \to \infty} \int_0^B \sin(x)\sin(x^2)\,dx$$

converges.

A–5 Three distinct points with integer coordinates lie in the plane on a circle of radius $r > 0$. Show that two of these points are separated by a distance of at least $r^{1/3}$.

A–6 Let $f(x)$ be a polynomial with integer coefficients. Define a sequence $a_0, a_1, \dots$ of integers such that $a_0 = 0$ and $a_{n+1} = f(a_n)$ for all $n \geq 0$. Prove that if there exists a positive integer m for which $a_m = 0$ then either $a_1 = 0$ or $a_2 = 0$.

B–1 Let a_j, b_j, c_j be integers for $1 \leq j \leq N$. Assume for each j, at least one of a_j, b_j, c_j is odd. Show that there exist integers r, s, t such that $ra_j + sb_j + tc_j$ is odd for at least $4N/7$ values of j, $1 \leq j \leq N$.

B–2 Prove that the expression

$$\frac{gcd(m,n)}{n}\binom{n}{m}$$

is an integer for all pairs of integers $n \geq m \geq 1$.

B–3 Let $f(t) = \sum_{j=1}^{N} a_j \sin(2\pi jt)$, where each a_j is real and a_N is not equal to 0. Let N_k denote the number of zeroes (including multiplicities) of $\frac{d^k f}{dt^k}$. Prove that

$$N_0 \leq N_1 \leq N_2 \leq \cdots \text{ and } \lim_{k \to \infty} N_k = 2N.$$

[Editorial clarification: only zeroes in $[0,1)$ should be counted.]

B–4 Let $f(x)$ be a continuous function such that $f(2x^2 - 1) = 2xf(x)$ for all x. Show that $f(x) = 0$ for $-1 \leq x \leq 1$.

B–5 Let S_0 be a finite set of positive integers. We define finite sets $S_1, S_2, \dots$ of positive integers as follows: the integer a is in S_{n+1} if and only if exactly one of $a-1$ or a is in S_n. Show that there exist infinitely many integers N for which $S_N = S_0 \cup \{N + a : a \in S_0\}$.

B–6 Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm1, \pm1, \dots, \pm1)$ in n-dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.