Solutions to the 61st William Lowell Putnam Mathematical Competition
Saturday, December 2, 2000

Manjul Bhargava, Kiran Kedlaya, and Lenny Ng

A–1 The possible values comprise the interval $(0, A^2)$.

To see that the values must lie in this interval, note that

$$\left(\sum_{j=0}^m x_j\right)^2 = \sum_{j=0}^m x_j^2 + \sum_{0 \leq j<k \leq m} 2x_j x_k,$$

so $\sum_{j=0}^m x_j^2 \leq A^2 - 2x_0x_1$. Letting $m \to \infty$, we have $\sum_{j=0}^\infty x_j^2 \leq A^2 - 2x_0x_1 < A^2$.

To show that all values in $(0, A^2)$ can be obtained, we use geometric progressions with $x_1/x_0 = x_2/x_1 = \cdots = d$ for variable d. Then $\sum_{j=0}^\infty x_j = x_0/(1-d)$ and

$$\sum_{j=0}^\infty x_j^2 = \frac{x_0^2}{1-d^2} = \frac{1-d}{1+d}\left(\sum_{j=0}^\infty x_j\right)^2.$$

As d increases from 0 to 1, $(1-d)/(1+d)$ decreases from 1 to 0. Thus if we take geometric progressions with $\sum_{j=0}^\infty x_j = A$, $\sum_{j=0}^\infty x_j^2$ ranges from 0 to A^2. Thus the possible values are indeed those in the interval $(0, A^2)$, as claimed.

A–2 First solution: Let a be an even integer such that a^2+1 is not prime. (For example, choose $a \equiv 2 \pmod 5$, so that a^2+1 is divisible by 5.) Then we can write a^2+1 as a difference of squares $x^2 - b^2$, by factoring a^2+1 as rs with $r \geq s > 1$, and setting $x = (r+s)/2$, $b = (r-s)/2$. Finally, put $n = x^2-1$, so that $n = a^2+b^2$, $n+1 = x^2$, $n+2 = x^2+1$.

Second solution: It is well-known that the equation $x^2 - 2y^2 = 1$ has infinitely many solutions (the so-called "Pell" equation). Thus setting $n = 2y^2$ (so that $n = y^2+y^2$, $n+1 = x^2+0^2$, $n+2 = x^2+1^2$) yields infinitely many n with the desired property.

Third solution: As in the first solution, it suffices to exhibit x such that x^2-1 is the sum of two squares. We will take $x = 3^{2^n}$, and show that x^2-1 is the sum of two squares by induction on n: if $3^{2^n} - 1 = a^2+b^2$, then

$$\begin{aligned}(3^{2^{n+1}} - 1) &= (3^{2^n} - 1)(3^{2^n} + 1) \\ &= (3^{2^{n-1}}a + b)^2 + (a - 3^{2^{n-1}}b)^2.\end{aligned}$$

Fourth solution (by Jonathan Weinstein): Let $n = 4k^4 + 4k^2 = (2k^2)^2 + (2k)^2$ for any integer k. Then $n+1 = (2k^2+1)^2 + 0^2$ and $n+2 = (2k^2+1)^2 + 1^2$.

A–3 The maximum area is $3\sqrt{5}$.

We deduce from the area of $P_1P_3P_5P_7$ that the radius of the circle is $\sqrt{5/2}$. An easy calculation using the Pythagorean Theorem then shows that the rectangle $P_2P_4P_6P_8$ has sides $\sqrt{2}$ and $2\sqrt{2}$. For notational ease, denote the area of a polygon by putting brackets around the name of the polygon.

By symmetry, the area of the octagon can be expressed as

$$[P_2P_4P_6P_8] + 2[P_2P_3P_4] + 2[P_4P_5P_6].$$

Note that $[P_2P_3P_4]$ is $\sqrt{2}$ times the distance from P_3 to P_2P_4, which is maximized when P_3 lies on the midpoint of arc P_2P_4; similarly, $[P_4P_5P_6]$ is $\sqrt{2}/2$ times the distance from P_5 to P_4P_6, which is maximized when P_5 lies on the midpoint of arc P_4P_6. Thus the area of the octagon is maximized when P_3 is the midpoint of arc P_2P_4 and P_5 is the midpoint of arc P_4P_6. In this case, it is easy to calculate that $[P_2P_3P_4] = \sqrt{5}-1$ and $[P_4P_5P_6] = \sqrt{5}/2 - 1$, and so the area of the octagon is $3\sqrt{5}$.

A–4 To avoid some improper integrals at 0, we may as well replace the left endpoint of integration by some $\varepsilon > 0$. We now use integration by parts:

$$\begin{aligned}\int_\varepsilon^B \sin x \sin x^2\,dx &= \int_\varepsilon^B \frac{\sin x}{2x} \sin x^2 (2x\,dx) \\ &= -\frac{\sin x}{2x}\cos x^2 \Big|_\varepsilon^B \\ &\quad + \int_\varepsilon^B \left(\frac{\cos x}{2x} - \frac{\sin x}{2x^2}\right)\cos x^2\,dx.\end{aligned}$$

Now $\frac{\sin x}{2x}\cos x^2$ tends to 0 as $B \to \infty$, and the integral of $\frac{\sin x}{2x^2}\cos x^2$ converges absolutely by comparison with $1/x^2$. Thus it suffices to note that

$$\begin{aligned}\int_\varepsilon^B \frac{\cos x}{2x}\cos x^2\,dx &= \int_\varepsilon^B \frac{\cos x}{4x^2}\cos x^2 (2x\,dx) \\ &= \frac{\cos x}{4x^2}\sin x^2\Big|_\varepsilon^B \\ &\quad - \int_\varepsilon^B \frac{2x\cos x - \sin x}{4x^3}\sin x^2\,dx,\end{aligned}$$

and that the final integral converges absolutely by comparison to $1/x^3$.

An alternate approach is to first rewrite $\sin x \sin x^2$ as $\frac{1}{2}(\cos(x^2-x) - \cos(x^2+x))$. Then

$$\begin{aligned}\int_\varepsilon^B \cos(x^2+x)\,dx &= -\frac{\sin(x^2+x)}{2x+1}\Big|_\varepsilon^B \\ &\quad - \int_\varepsilon^B \frac{2\sin(x^2+x)}{(2x+1)^2}\,dx\end{aligned}$$

converges absolutely, and $\int_0^B \cos(x^2-x)$ can be treated similarly.

A–5 Let a,b,c be the distances between the points. Then the area of the triangle with the three points as vertices is $abc/4r$. On the other hand, the area of a triangle whose vertices have integer coordinates is at least 1/2 (for example, by Pick's Theorem). Thus $abc/4r \geq 1/2$, and so

$$\max\{a,b,c\} \geq (abc)^{1/3} \geq (2r)^{1/3} > r^{1/3}.$$

A–6 Recall that if $f(x)$ is a polynomial with integer coefficients, then $m-n$ divides $f(m)-f(n)$ for any integers m and n. In particular, if we put $b_n = a_{n+1}-a_n$, then b_n divides b_{n+1} for all n. On the other hand, we are given that $a_0 = a_m = 0$, which implies that $a_1 = a_{m+1}$ and so $b_0 = b_m$. If $b_0 = 0$, then $a_0 = a_1 = \cdots = a_m$ and we are done. Otherwise, $|b_0| = |b_1| = |b_2| = \cdots$, so $b_n = \pm b_0$ for all n.

Now $b_0 + \cdots + b_{m-1} = a_m - a_0 = 0$, so half of the integers $b_0, \dots, b_{m-1}$ are positive and half are negative. In particular, there exists an integer $0 < k < m$ such that $b_{k-1} = -b_k$, which is to say, $a_{k-1} = a_{k+1}$. From this it follows that $a_n = a_{n+2}$ for all $n \geq k-1$; in particular, for $m = n$, we have

$$a_0 = a_m = a_{m+2} = f(f(a_0)) = a_2.$$

B–1 Consider the seven triples (a,b,c) with $a,b,c \in \{0,1\}$ not all zero. Notice that if r_j, s_j, t_j are not all even, then four of the sums $ar_j + bs_j + ct_j$ with $a,b,c \in \{0,1\}$ are even and four are odd. Of course the sum with $a = b = c = 0$ is even, so at least four of the seven triples with a,b,c not all zero yield an odd sum. In other words, at least $4N$ of the tuples (a,b,c,j) yield odd sums. By the pigeonhole principle, there is a triple (a,b,c) for which at least $4N/7$ of the sums are odd.

B–2 Since $\gcd(m,n)$ is an integer linear combination of m and n, it follows that

$$\frac{gcd(m,n)}{n}\binom{n}{m}$$

is an integer linear combination of the integers

$$\frac{m}{n}\binom{n}{m} = \binom{n-1}{m-1} \text{ and } \frac{n}{n}\binom{n}{m} = \binom{n}{m}$$

and hence is itself an integer.

B–3 Put $f_k(t) = \frac{df^k}{dt^k}$. Recall Rolle's theorem: if $f(t)$ is differentiable, then between any two zeroes of $f(t)$ there exists a zero of $f'(t)$. This also applies when the zeroes are not all distinct: if f has a zero of multiplicity m at $t = x$, then f' has a zero of multiplicity at least $m-1$ there.

Therefore, if $0 \leq a_0 \leq a_1 \leq \cdots \leq a_r < 1$ are the roots of f_k in $[0,1)$, then f_{k+1} has a root in each of the intervals $(a_0,a_1), (a_1,a_2), \dots, (a_{r-1},a_r)$, so long as we adopt the convention that the empty interval (t,t) actually contains the point t itself. There is also a root in the "wraparound" interval (a_r, a_0). Thus $N_{k+1} \geq N_k$.

Next, note that if we set $z = e^{2\pi i t}$; then

$$f_{4k}(t) = \frac{1}{2i}\sum_{j=1}^N j^{4k} a_j (z^j - z^{-j})$$

is equal to z^{-N} times a polynomial of degree $2N$. Hence as a function of z, it has at most $2N$ roots; therefore $f_k(t)$ has at most $2N$ roots in $[0,1]$. That is, $N_k \leq 2N$ for all N.

To establish that $N_k \to 2N$, we make precise the observation that

$$f_k(t) = \sum_{j=1}^N j^{4k} a_j \sin(2\pi j t)$$

is dominated by the term with $j = N$. At the points $t = (2i+1)/(2N)$ for $i = 0,1,\dots,N-1$, we have $N^{4k} a_N \sin(2\pi N t) = \pm N^{4k} a_N$. If k is chosen large enough so that

$$|a_N| N^{4k} > |a_1| 1^{4k} + \cdots + |a_{N-1}| (N-1)^{4k},$$

then $f_k((2i+1)/2N)$ has the same sign as $a_N \sin(2\pi N a t)$, which is to say, the sequence $f_k(1/2N), f_k(3/2N), \dots$ alternates in sign. Thus between these points (again including the "wraparound" interval) we find $2N$ sign changes of f_k. Therefore $\lim_{k\to\infty} N_k = 2N$.

B–4 For t real and not a multiple of π, write $g(t) = \frac{f(\cos t)}{\sin t}$. Then $g(t+\pi) = g(t)$; furthermore, the given equation implies that

$$g(2t) = \frac{f(2\cos^2 t - 1)}{\sin(2t)} = \frac{2(\cos t) f(\cos t)}{\sin(2t)} = g(t).$$

In particular, for any integer n and k, we have

$$g(1 + n\pi/2^k) = g(2^k + n\pi) = g(2^k) = g(1).$$

Since f is continuous, g is continuous where it is defined; but the set $\{1 + n\pi/2^k | n,k \in \mathbb{Z}\}$ is dense in the reals, and so g must be constant on its domain. Since $g(-t) = -g(t)$ for all t, we must have $g(t) = 0$ when t is not a multiple of π. Hence $f(x) = 0$ for $x \in (-1,1)$. Finally, setting $x = 0$ and $x = 1$ in the given equation yields $f(-1) = f(1) = 0$.

B–5 We claim that all integers N of the form 2^k, with k a positive integer and $N > \max\{S_0\}$, satisfy the desired conditions.

It follows from the definition of S_n, and induction on n, that

$$\begin{aligned}\sum_{j\in S_n} x^j &\equiv (1+x)\sum_{j\in S_{n-1}} x^j \\ &\equiv (1+x)^n \sum_{j\in S_0} x^j \pmod{2}.\end{aligned}$$

From the identity $(x+y)^2 \equiv x^2+y^2 \pmod{2}$ and induction on n, we have $(x+y)^{2^n} \equiv x^{2^n}+y^{2^n} \pmod{2}$. Hence if we choose N to be a power of 2 greater than $\max\{S_0\}$, then

$$\sum_{j\in S_n} \equiv (1+x^N)\sum_{j\in S_0} x^j$$

and $S_N = S_0 \cup \{N+a : a \in S_0\}$, as desired.

B–6 For each point P in B, let S_P be the set of points with all coordinates equal to ± 1 which differ from P in exactly one coordinate. Since there are more than $2^{n+1}/n$ points in B, and each S_P has n elements, the cardinalities of the sets S_P add up to more than 2^{n+1}, which is to say, more than twice the total number of points. By the pigeonhole principle, there must be a point in three of the sets, say S_P, S_Q, S_R. But then any two of P, Q, R differ in exactly two coordinates, so PQR is an equilateral triangle, as desired.