# The 67th William Lowell Putnam Mathematical Competition
## Saturday, December 2, 2006

A–1 Find the volume of the region of points $(x,y,z)$ such that

$$(x^2+y^2+z^2+8)^2 \leq 36(x^2+y^2).$$

A–2 Alice and Bob play a game in which they take turns removing stones from a heap that initially has $n$ stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many $n$ such that Bob has a winning strategy. (For example, if $n=17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

A–3 Let $1,2,3,\dots,2005,2006,2007,2009,2012,2016,\dots$ be a sequence defined by $x_k = k$ for $k=1,2,\dots,2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

A–4 Let $S=\{1,2,\dots,n\}$ for some integer $n>1$. Say a permutation $\pi$ of $S$ has a *local maximum* at $k \in S$ if

(i) $\pi(k) > \pi(k+1)$ for $k=1$;

(ii) $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1<k<n$;

(iii) $\pi(k-1) < \pi(k)$ for $k=n$.

(For example, if $n=5$ and $\pi$ takes values at $1,2,3,4,5$ of $2,1,4,5,3$, then $\pi$ has a local maximum of 2 at $k=1$, and a local maximum of 5 at $k=4$.) What is the average number of local maxima of a permutation of $S$, averaging over all permutations of $S$?

A–5 Let $n$ be a positive odd integer and let $\theta$ be a real number such that $\theta/\pi$ is irrational. Set $a_k = \tan(\theta + k\pi/n)$, $k=1,2,\dots,n$. Prove that

$$\frac{a_1+a_2+\cdots+a_n}{a_1 a_2 \cdots a_n}$$

is an integer, and determine its value.

A–6 Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

B–1 Show that the curve $x^3+3xy+y^3=1$ contains only one set of three distinct points, $A$, $B$, and $C$, which are vertices of an equilateral triangle, and find its area.

B–2 Prove that, for every set $X=\{x_1,x_2,\dots,x_n\}$ of $n$ real numbers, there exists a non-empty subset $S$ of $X$ and an integer $m$ such that

$$\left| m + \sum_{s\in S} s \right| \leq \frac{1}{n+1}.$$

B–3 Let $S$ be a finite set of points in the plane. A linear partition of $S$ is an unordered pair $\{A,B\}$ of subsets of $S$ such that $A\cup B = S$, $A\cap B = \emptyset$, and $A$ and $B$ lie on opposite sides of some straight line disjoint from $S$ ($A$ or $B$ may be empty). Let $L_S$ be the number of linear partitions of $S$. For each positive integer $n$, find the maximum of $L_S$ over all sets $S$ of $n$ points.

B–4 Let $Z$ denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus $Z$ has $2^n$ elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Given a vector subspace $V$ of $\mathbb{R}^n$, let $Z(V)$ denote the number of members of $Z$ that lie in $V$. Let $k$ be given, $0\leq k\leq n$. Find the maximum, over all vector subspaces $V\subseteq \mathbb{R}^n$ of dimension $k$, of the number of points in $V\cap Z$. [Editorial note: the proposers probably intended to write $Z(V)$ instead of "the number of points in $V\cap Z$", but this changes nothing.]

B–5 For each continuous function $f:[0,1]\to\mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)\,dx$ and $J(x) = \int_0^1 x\,(f(x))^2\,dx$. Find the maximum value of $I(f)-J(f)$ over all such functions $f$.

B–6 Let $k$ be an integer greater than 1. Suppose $a_0>0$, and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n>0$. Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$