The 70th William Lowell Putnam Mathematical Competition
Saturday, December 5, 2009

A1 Let f be a real-valued function on the plane such that for every square $ABCD$ in the plane, $f(A)+f(B)+f(C)+f(D)=0$. Does it follow that $f(P)=0$ for all points P in the plane?

A2 Functions f,g,h are differentiable on some open interval around 0 and satisfy the equations and initial conditions

$$\begin{aligned} f' &= 2f^2gh + \frac{1}{gh}, \quad f(0)=1, \\ g' &= fg^2h + \frac{4}{fh}, \quad g(0)=1, \\ h' &= 3fgh^2 + \frac{1}{fg}, \quad h(0)=1. \end{aligned}$$

Find an explicit formula for $f(x)$, valid in some open interval around 0.

A3 Let d_n be the determinant of the $n \times n$ matrix whose entries, from left to right and then from top to bottom, are $\cos 1, \cos 2, \dots, \cos n^2$. (For example,

$$d_3 = \begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}.$$

The argument of cos is always in radians, not degrees.) Evaluate $\lim_{n\to\infty} d_n$.

A4 Let S be a set of rational numbers such that

(a) $0 \in S$;

(b) If $x \in S$ then $x+1 \in S$ and $x-1 \in S$; and

(c) If $x \in S$ and $x \notin \{0,1\}$, then $\frac{1}{x(x-1)} \in S$.

Must S contain all rational numbers?

A5 Is there a finite abelian group G such that the product of the orders of all its elements is 2^{2009}?

A6 Let $f : [0,1]^2 \to \mathbb{R}$ be a continuous function on the closed unit square such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous on the interior $(0,1)^2$. Let $a = \int_0^1 f(0,y)\,dy$, $b = \int_0^1 f(1,y)\,dy$, $c = \int_0^1 f(x,0)\,dx$, $d = \int_0^1 f(x,1)\,dx$. Prove or disprove: There must be a point (x_0,y_0) in $(0,1)^2$ such that

$$\frac{\partial f}{\partial x}(x_0,y_0) = b-a \quad \text{and} \quad \frac{\partial f}{\partial y}(x_0,y_0) = d-c.$$

B1 Show that every positive rational number can be written as a quotient of products of factorials of (not necessarily distinct) primes. For example,

$$\frac{10}{9} = \frac{2!\cdot 5!}{3!\cdot 3!\cdot 3!}.$$

B2 A game involves jumping to the right on the real number line. If a and b are real numbers and $b > a$, the cost of jumping from a to b is $b^3 - ab^2$. For what real numbers c can one travel from 0 to 1 in a finite number of jumps with total cost exactly c?

B3 Call a subset S of $\{1,2,\dots,n\}$ *mediocre* if it has the following property: Whenever a and b are elements of S whose average is an integer, that average is also an element of S. Let $A(n)$ be the number of mediocre subsets of $\{1,2,\dots,n\}$. [For instance, every subset of $\{1,2,3\}$ except $\{1,3\}$ is mediocre, so $A(3)=7$.] Find all positive integers n such that $A(n+2)-2A(n+1)+A(n)=1$.

B4 Say that a polynomial with real coefficients in two variables, x,y, is *balanced* if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space V over $\mathbb{R}$. Find the dimension of V.

B5 Let $f : (1,\infty) \to \mathbb{R}$ be a differentiable function such that

$$f'(x) = \frac{x^2 - f(x)^2}{x^2(f(x)^2+1)} \qquad \text{for all } x > 1.$$

Prove that $\lim_{x\to\infty} f(x) = \infty$.

B6 Prove that for every positive integer n, there is a sequence of integers $a_0, a_1, \dots, a_{2009}$ with $a_0 = 0$ and $a_{2009} = n$ such that each term after a_0 is either an earlier term plus 2^k for some nonnegative integer k, or of the form $b \bmod c$ for some earlier positive terms b and c. [Here $b \bmod c$ denotes the remainder when b is divided by c, so $0 \le (b \bmod c) < c$.]