The 75th William Lowell Putnam Mathematical Competition
Saturday, December 6, 2014

A1 Prove that every nonzero coefficient of the Taylor series of

$$(1-x+x^2)e^x$$

about $x=0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

A2 Let A be the $n\times n$ matrix whose entry in the i-th row and j-th column is

$$\frac{1}{\min(i,j)}$$

for $1\le i,j\le n$. Compute $\det(A)$.

A3 Let $a_0=5/2$ and $a_k=a_{k-1}^2-2$ for $k\ge 1$. Compute

$$\prod_{k=0}^{\infty}\left(1-\frac{1}{a_k}\right)$$

in closed form.

A4 Suppose X is a random variable that takes on only nonnegative integer values, with $E[X]=1$, $E\left[X^2\right]=2$, and $E\left[X^3\right]=5$. (Here $E[y]$ denotes the expectation of the random variable Y.) Determine the smallest possible value of the probability of the event $X=0$.

A5 Let

$$P_n(x)=1+2x+3x^2+\cdots+nx^{n-1}.$$

Prove that the polynomials $P_j(x)$ and $P_k(x)$ are relatively prime for all positive integers j and k with $j\neq k$.

A6 Let n be a positive integer. What is the largest k for which there exist $n\times n$ matrices $M_1,\dots,M_k$ and $N_1,\dots,N_k$ with real entries such that for all i and j, the matrix product M_iN_j has a zero entry somewhere on its diagonal if and only if $i\neq j$?

B1 A *base* 10 *over-expansion* of a positive integer N is an expression of the form

$$N=d_k10^k+d_{k-1}10^{k-1}+\cdots+d_010^0$$

with $d_k\neq 0$ and $d_i\in\{0,1,2,\dots,10\}$ for all i. For instance, the integer $N=10$ has two base 10 over-expansions: $10=10\cdot 10^0$ and the usual base 10 expansion $10=1\cdot 10^1+0\cdot 10^0$. Which positive integers have a unique base 10 over-expansion?

B2 Suppose that f is a function on the interval $[1,3]$ such that $-1\le f(x)\le 1$ for all x and $\int_1^3 f(x)\,dx=0$. How large can $\int_1^3\frac{f(x)}{x}\,dx$ be?

B3 Let A be an $m\times n$ matrix with rational entries. Suppose that there are at least $m+n$ distinct prime numbers among the absolute values of the entries of A. Show that the rank of A is at least 2.

B4 Show that for each positive integer n, all the roots of the polynomial

$$\sum_{k=0}^{n}2^{k(n-k)}x^k$$

are real numbers.

B5 In the 75th annual Putnam Games, participants compete at mathematical games. Patniss and Keeta play a game in which they take turns choosing an element from the group of invertible $n\times n$ matrices with entries in the field $\mathbb{Z}/p\mathbb{Z}$ of integers modulo p, where n is a fixed positive integer and p is a fixed prime number. The rules of the game are:

(1) A player cannot choose an element that has been chosen by either player on any previous turn.

(2) A player can only choose an element that commutes with all previously chosen elements.

(3) A player who cannot choose an element on his/her turn loses the game.

Patniss takes the first turn. Which player has a winning strategy? (Your answer may depend on n and p.)

B6 Let $f:[0,1]\to\mathbb{R}$ be a function for which there exists a constant $K>0$ such that $|f(x)-f(y)|\le K|x-y|$ for all $x,y\in[0,1]$. Suppose also that for each rational number $r\in[0,1]$, there exist integers a and b such that $f(r)=a+br$. Prove that there exist finitely many intervals $I_1,\dots,I_n$ such that f is a linear function on each I_i and $[0,1]=\bigcup_{i=1}^n I_i$.