The 80th William Lowell Putnam Mathematical Competition
Saturday, December 7, 2019

A1 Determine all possible values of the expression

$$A^3+B^3+C^3-3ABC$$

where A, B, and C are nonnegative integers.

A2 In the triangle $\triangle ABC$, let G be the centroid, and let I be the center of the inscribed circle. Let α and β be the angles at the vertices A and B, respectively. Suppose that the segment IG is parallel to AB and that $\beta = 2\tan^{-1}(1/3)$. Find α.

A3 Given real numbers $b_0, b_1, \dots, b_{2019}$ with $b_{2019} \neq 0$, let $z_1, z_2, \dots, z_{2019}$ be the roots in the complex plane of the polynomial

$$P(z) = \sum_{k=0}^{2019} b_k z^k.$$

Let $\mu = (|z_1| + \cdots + |z_{2019}|)/2019$ be the average of the distances from $z_1, z_2, \dots, z_{2019}$ to the origin. Determine the largest constant M such that $\mu \geq M$ for all choices of $b_0, b_1, \dots, b_{2019}$ that satisfy

$$1 \leq b_0 < b_1 < b_2 < \cdots < b_{2019} \leq 2019.$$

A4 Let f be a continuous real-valued function on $\mathbb{R}^3$. Suppose that for every sphere S of radius 1, the integral of $f(x,y,z)$ over the surface of S equals 0. Must $f(x,y,z)$ be identically 0?

A5 Let p be an odd prime number, and let $\mathbb{F}_p$ denote the field of integers modulo p. Let $\mathbb{F}_p[x]$ be the ring of polynomials over $\mathbb{F}_p$, and let $q(x) \in \mathbb{F}_p[x]$ be given by

$$q(x) = \sum_{k=1}^{p-1} a_k x^k,$$

where

$$a_k = k^{(p-1)/2} \mod p.$$

Find the greatest nonnegative integer n such that $(x-1)^n$ divides $q(x)$ in $\mathbb{F}_p[x]$.

A6 Let g be a real-valued function that is continuous on the closed interval $[0,1]$ and twice differentiable on the open interval $(0,1)$. Suppose that for some real number $r > 1$,

$$\lim_{x \to 0^+} \frac{g(x)}{x^r} = 0.$$

Prove that either

$$\lim_{x \to 0^+} g'(x) = 0 \qquad \text{or} \qquad \limsup_{x \to 0^+} x^r |g''(x)| = \infty.$$

B1 Denote by $\mathbb{Z}^2$ the set of all points (x,y) in the plane with integer coordinates. For each integer $n \geq 0$, let P_n be the subset of $\mathbb{Z}^2$ consisting of the point $(0,0)$ together with all points (x,y) such that $x^2+y^2 = 2^k$ for some integer $k \leq n$. Determine, as a function of n, the number of four-point subsets of P_n whose elements are the vertices of a square.

B2 For all $n \geq 1$, let

$$a_n = \sum_{k=1}^{n-1} \frac{\sin\left(\frac{(2k-1)\pi}{2n}\right)}{\cos^2\left(\frac{(k-1)\pi}{2n}\right)\cos^2\left(\frac{k\pi}{2n}\right)}.$$

Determine

$$\lim_{n \to \infty} \frac{a_n}{n^3}.$$

B3 Let Q be an n-by-n real orthogonal matrix, and let $u \in \mathbb{R}^n$ be a unit column vector (that is, $u^T u = 1$). Let $P = I - 2uu^T$, where I is the n-by-n identity matrix. Show that if 1 is not an eigenvalue of Q, then 1 is an eigenvalue of PQ.

B4 Let $\mathscr{F}$ be the set of functions $f(x,y)$ that are twice continuously differentiable for $x \geq 1$, $y \geq 1$ and that satisfy the following two equations (where subscripts denote partial derivatives):

$$xf_x + yf_y = xy\ln(xy),$$
$$x^2 f_{xx} + y^2 f_{yy} = xy.$$

For each $f \in \mathscr{F}$, let

$$m(f) = \min_{s \geq 1}\left(f(s+1,s+1) - f(s+1,s) - f(s,s+1) + f(s,s)\right).$$

Determine $m(f)$, and show that it is independent of the choice of f.

B5 Let F_m be the mth Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_m = F_{m-1} + F_{m-2}$ for all $m \geq 3$. Let $p(x)$ be the polynomial of degree 1008 such that $p(2n+1) = F_{2n+1}$ for $n = 0, 1, 2, \dots, 1008$. Find integers j and k such that $p(2019) = F_j - F_k$.

B6 Let $\mathbb{Z}^n$ be the integer lattice in $\mathbb{R}^n$. Two points in $\mathbb{Z}^n$ are called *neighbors* if they differ by exactly 1 in one coordinate and are equal in all other coordinates. For which integers $n \geq 1$ does there exist a set of points $S \subset \mathbb{Z}^n$ satisfying the following two conditions?

(1) If p is in S, then none of the neighbors of p is in S.

(2) If $p \in \mathbb{Z}^n$ is not in S, then exactly one of the neighbors of p is in S.