A1 Determine all possible values of the expression
\[ A^3 + B^3 + C^3 - 3ABC \]
where \( A, B, \) and \( C \) are nonnegative integers.

A2 In the triangle \( \triangle ABC \), let \( G \) be the centroid, and let \( I \) be the center of the inscribed circle. Let \( \alpha \) and \( \beta \) be the angles at the vertices \( A \) and \( B \), respectively. Suppose that the segment \( IG \) is parallel to \( AB \) and that \( \beta = \frac{2 \tan^{-1}(1/3)}{1} \). Find \( \alpha \).

A3 Given real numbers \( b_0, b_1, \ldots, b_{2019} \) with \( b_{2019} \neq 0 \), let \( z_1, z_2, \ldots, z_{2019} \) be the roots in the complex plane of the polynomial
\[ P(z) = \sum_{k=0}^{2019} b_k z^k. \]
Let \( \mu = (|z_1| + \cdots + |z_{2019}|)/2019 \) be the average of the distances from \( z_1, z_2, \ldots, z_{2019} \) to the origin. Determine the largest constant \( M \) such that \( \mu \geq M \) for all choices of \( b_0, b_1, \ldots, b_{2019} \) that satisfy
\[ 1 \leq b_0 < b_1 < b_2 < \cdots < b_{2019} \leq 2019. \]

A4 Let \( f \) be a continuous real-valued function on \( \mathbb{R}^3 \). Suppose that for every sphere \( S \) of radius 1, the integral of \( f(x, y, z) \) over the surface of \( S \) equals 0. Must \( f(x, y, z) \) be identically 0?

A5 Let \( p \) be an odd prime number, and let \( \mathbb{F}_p \) denote the field of integers modulo \( p \). Let \( \mathbb{F}_p[x] \) be the ring of polynomials over \( \mathbb{F}_p \), and let \( q(x) \in \mathbb{F}_p[x] \) be given by
\[ q(x) = \sum_{k=1}^{p-1} a_k x^k, \]
where
\[ a_k = k^{(p-1)/2} \mod p. \]
Find the greatest nonnegative integer \( n \) such that \((x - 1)^n \) divides \( q(x) \) in \( \mathbb{F}_p[x] \).

A6 Let \( g \) be a real-valued function that is continuous on the closed interval \([0, 1]\) and twice differentiable on the open interval \((0, 1)\). Suppose that for some real number \( r > 1 \),
\[ \lim_{x \to 0^+} \frac{g(x)}{x^r} = 0. \]
Prove that either
\[ \lim_{x \to 0^+} g'(x) = 0 \quad \text{or} \quad \limsup_{x \to 0^+} |g''(x)| = \infty. \]

B1 Denote by \( \mathbb{Z}^2 \) the set of all points \((x, y)\) in the plane with integer coordinates. For each integer \( n \geq 0 \), let \( P_n \) be the subset of \( \mathbb{Z}^2 \) consisting of the points \((0, 0)\) together with all points \((x, y)\) such that \( x^2 + y^2 = 2^k \) for some integer \( k \leq n \). Determine, as a function of \( n \), the number of four-point subsets of \( P_n \) whose elements are the vertices of a square.

B2 For all \( n \geq 1 \), let
\[ a_n = \sum_{k=1}^{n-1} \sin \left( \frac{(2k-1)\pi}{2n} \right) \frac{k}{\cos^2 \left( \frac{(k-1)\pi}{2n} \right) \cos^2 \left( \frac{k\pi}{2n} \right)}. \]
Determine
\[ \lim_{n \to \infty} \frac{a_n}{n^3}. \]

B3 Let \( Q \) be an \( n \times n \) real orthogonal matrix, and let \( u \in \mathbb{R}^n \) be a unit column vector (that is, \( u^T u = 1 \)). Let \( P = I - 2uu^T \), where \( I \) is the \( n \times n \) identity matrix. Show that if 1 is not an eigenvalue of \( Q \), then 1 is an eigenvalue of \( PQ \).

B4 Let \( F \) be the set of functions \( f(x, y) \) that are twice continuously differentiable for \( x \geq 1, y \geq 1 \) and that satisfy the following two equations (where subscripts denote partial derivatives):
\[ \begin{align*}
xf_x + yf_y &= xy \ln(xy), \\
x^2f_{xx} + y^2f_{yy} &= xy.
\end{align*} \]
For each \( f \in F \), let
\[ m(f) = \min_{x \geq 1} (f(s + 1, s + 1) - f(s + 1, s) - f(s, s + 1) + f(s, s)). \]
Determine \( m(f) \), and show that it is independent of the choice of \( f \).

B5 Let \( F_m \) be the \( m \)th Fibonacci number, defined by \( F_1 = F_2 = 1 \) and \( F_m = F_{m-1} + F_{m-2} \) for all \( m \geq 3 \). Let \( p(x) \) be the polynomial of degree 1008 such that \( p(2n+1) = F_{2n+1} \) for \( n = 0, 1, 2, \ldots, 1008 \). Find integers \( j \) and \( k \) such that \( p(2019) = F_j - F_k \).

B6 Let \( \mathbb{Z}^n \) be the integer lattice in \( \mathbb{R}^n \). Two points in \( \mathbb{Z}^n \) are called neighbors if they differ by exactly 1 in one coordinate and are equal in all other coordinates. For which integers \( n \geq 1 \) does there exist a set of points \( S \subset \mathbb{Z}^n \) satisfying the following two conditions?

1. If \( p \) is in \( S \), then none of the neighbors of \( p \) is in \( S \).
2. If \( p \in \mathbb{Z}^n \) is not in \( S \), then exactly one of the neighbors of \( p \) is in \( S \).