The 85th William Lowell Putnam Mathematical Competition
Saturday, December 7, 2024

A1 Determine all positive integers n for which there exist positive integers a, b, and c satisfying
$$2a^n + 3b^n = 4c^n.$$

A2 For which real polynomials p is there a real polynomial q such that
$$p(p(x)) - x = (p(x) - x)^2 q(x)$$
for all real x?

A3 Let S be the set of bijections
$$T\colon \{1,2,3\} \times \{1,2,\dots,2024\} \to \{1,2,\dots,6072\}$$
such that $T(1,j) < T(2,j) < T(3,j)$ for all $j \in \{1,2,\dots,2024\}$ and $T(i,j) < T(i,j+1)$ for all $i \in \{1,2,3\}$ and $j \in \{1,2,\dots,2023\}$. Do there exist a and c in $\{1,2,3\}$ and b and d in $\{1,2,\dots,2024\}$ such that the fraction of elements T in S for which $T(a,b) < T(c,d)$ is at least $1/3$ and at most $2/3$?

A4 Find all primes $p > 5$ for which there exists an integer a and an integer r satisfying $1 \leq r \leq p-1$ with the following property: the sequence $1, a, a^2, \dots, a^{p-5}$ can be rearranged to form a sequence $b_0, b_1, b_2, \dots, b_{p-5}$ such that $b_n - b_{n-1} - r$ is divisible by p for $1 \leq n \leq p-5$.

A5 Consider a circle Ω with radius 9 and center at the origin $(0,0)$, and a disc Δ with radius 1 and center at $(r,0)$, where $0 \leq r \leq 8$. Two points P and Q are chosen independently and uniformly at random on Ω. Which value(s) of r minimize the probability that the chord $\overline{PQ}$ intersects Δ?

A6 Let $c_0, c_1, c_2, \dots$ be the sequence defined so that
$$\frac{1-3x-\sqrt{1-14x+9x^2}}{4} = \sum_{k=0}^{\infty} c_k x^k$$
for sufficiently small x. For a positive integer n, let A be the n-by-n matrix with i,j-entry c_{i+j-1} for i and j in $\{1,\dots,n\}$. Find the determinant of A.

B1 Let n and k be positive integers. The square in the ith row and jth column of an n-by-n grid contains the number $i+j-k$. For which n and k is it possible to select n squares from the grid, no two in the same row or column, such that the numbers contained in the selected squares are exactly $1,2,\dots,n$?

B2 Two convex quadrilaterals are called *partners* if they have three vertices in common and they can be labeled $ABCD$ and $ABCE$ so that E is the reflection of D across the perpendicular bisector of the diagonal $\overline{AC}$. Is there an infinite sequence of convex quadrilaterals such that each quadrilateral is a partner of its successor and no two elements of the sequence are congruent? [A diagram has been omitted.]

B3 Let r_n be the nth smallest positive solution to $\tan x = x$, where the argument of tangent is in radians. Prove that
$$0 < r_{n+1} - r_n - \pi < \frac{1}{(n^2+n)\pi}$$
for $n \geq 1$.

B4 Let n be a positive integer. Set $a_{n,0} = 1$. For $k \geq 0$, choose an integer $m_{n,k}$ uniformly at random from the set $\{1,\dots,n\}$, and let
$$a_{n,k+1} = \begin{cases} a_{n,k}+1, & \text{if } m_{n,k} > a_{n,k}; \\ a_{n,k}, & \text{if } m_{n,k} = a_{n,k}; \\ a_{n,k}-1, & \text{if } m_{n,k} < a_{n,k}. \end{cases}$$
Let $E(n)$ be the expected value of $a_{n,n}$. Determine $\lim_{n\to\infty} E(n)/n$.

B5 Let k and m be positive integers. For a positive integer n, let $f(n)$ be the number of integer sequences $x_1,\dots,x_k,y_1,\dots,y_m,z$ satisfying $1 \leq x_1 \leq \cdots \leq x_k \leq z \leq n$ and $1 \leq y_1 \leq \cdots \leq y_m \leq z \leq n$. Show that $f(n)$ can be expressed as a polynomial in n with nonnegative coefficients.

B6 For a real number a, let $F_a(x) = \sum_{n\geq 1} n^a e^{2n} x^{n^2}$ for $0 \leq x < 1$. Find a real number c such that
$$\lim_{x\to 1^-} F_a(x) e^{-1/(1-x)} = 0 \qquad \text{for all } a < c, \text{ and}$$
$$\lim_{x\to 1^-} F_a(x) e^{-1/(1-x)} = \infty \qquad \text{for all } a > c.$$