Sato-Tate groups of abelian surfaces and threefolds

Kiran S. Kedlaya
joint work (in progress) with Francesc Fité and Andrew V. Sutherland

Department of Mathematics, University of California, San Diego
School of Mathematics, Institute for Advanced Study (2018–2019)
kedlaya@ucsd.edu
http://kskedlaya.org/slides/

Arithmetic, Geometry, Cryptography and Coding Theory (AGC^{2}T-17)
Centre International de Rencontres Mathématiques, Luminy
June 11, 2019

Kedlaya was supported by NSF (grant DMS-1802161 and prior), UC San Diego (Warschawski Professorship), and IAS (Visiting Professorship). Fité was supported by IAS (NSF grant DMS-1638352). Sutherland was supported by NSF (grant DMS-1522526 and prior) and the Simons Collaboration on Arithmetic Geometry, Number Theory, and Computation, and received in-kind contributions from Google Cloud Platform.

Thanks to the organizers for the invitation

```
From: Alexey Zykin <alzykin@gmail.com>
Date: Thu, 2 Mar 2017 16:29:34 -1000
...

Dear Kiran,

I am writing you on behalf of the organizers of the
conference "Arithmetic, Geometry, Cryptography, and
Coding Theory" (AGCT-17) which is to take place in
May-June 2019 in Marseille, CIRM. Would you agree to
come as an invited speaker to the conference ?
...
```

Zeta functions of algebraic varieties

For X an algebraic variety over a finite field $\mathbb{F}_q$, the **zeta function** of X is

$$Z(X,T) = \prod_{x \in X^\circ} (1 - T^{\deg(x/\mathbb{F}_q)})^{-1} = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} \#X(\mathbb{F}_{q^n})\right),$$

where X° denotes the closed points of X (i.e., Galois orbits of $\overline{\mathbb{F}}_q$-points).

For X smooth proper over $\mathbb{F}_q$, we have

$$Z(X,T) = \frac{P_1(T) \cdots P_{2g-1}(T)}{P_0(T) \cdots P_{2g}(T)}$$

where $P_i(T)$ is (the reverse of) a q^i-**Weil polynomial**:

- $P_i(T)$ has integer coefficients and its constant term is 1.
- The roots of $P_i(T)$ in $\mathbb{C}$ all lie on the circle $|T| = q^{-i/2}$.

Curves and abelian varieties

When X is a (smooth, proper, geometrically integral) curve of genus g,

$$P_0(T) = 1 - T, \qquad P_2(T) = 1 - qT,$$

$P_1(T)$ is of degree $2g$, and $P_1(q^{-1/2}T)$ is palindromic.

When X is an abelian variety of dimension g, $P_1(T)$ is of degree $2g$, $P_1(q^{-1/2}T)$ is palindromic, and $P_i(T) = \wedge^i P_1(T)$. That is, if P_1 has roots $\alpha_1, \dots, \alpha_{2g}$, then P_i has roots

$$\alpha_{j_1} \cdots \alpha_{j_i} \qquad (1 \leq j_1 < \cdots < j_i \leq 2g).$$

The values of P_1 for a curve and its Jacobian coincide.

L-functions

For A an abelian variety over a number field K with ring of integers $\mathfrak{o}_K$, its **(incomplete)** L**-function** is the Dirichlet series

$$L(A,s) = \prod_{\mathfrak{p}} L_{\mathfrak{p}}(\mathrm{Norm}(\mathfrak{p})^{-s})^{-1}$$

where $\mathfrak{p}$ runs over prime ideals of $\mathfrak{o}_K$ at which A has good reduction, $\mathrm{Norm}(\mathfrak{p}) = \#(\mathfrak{o}_K/\mathfrak{p})$ is the absolute norm, and $L_{\mathfrak{p}}(T)$ is the factor $P_1(T)$ of the zeta function of the reduction of A modulo $\mathfrak{p}$.

For example, if A is an elliptic curve over $\mathbb{Q}$, this is the usual expression

$$L(A,s) = \prod_{p} (1 - a_p p^{-s} + p^{1-2s})^{-1}, \qquad a_p = p + 1 - \#A(\mathbb{F}_p).$$

In general, $L(A,s)$ converges absolutely for $\mathrm{Re}(s) > 3/2$ but is expected to admit a meromorphic continuation to $\mathbb{C}$ (more on this later).

Distribution of Euler factors

With the functional equation in mind, we renormalize the L-polynomials:

$$\overline{L}_{\mathfrak{p}}(T) = L_{\mathfrak{p}}(\mathrm{Norm}(\mathfrak{p})^{-1/2}T) = 1 + a_1 T + \cdots + a_{2g-1}T^{2g-1} + T^{2g}.$$

This polynomial is determined by the point $(a_1, \dots, a_g)$ which lies in a bounded region of $\mathbb{R}^g$. It is natural to ask whether these points admit a limiting distribution as $\mathfrak{p}$ varies, and if so what this can be.

For E/K an elliptic curve, there are conjecturally* 3 possible distributions, each corresponding to traces of random matrices:

- one when E has CM defined over K (matrices in $\mathrm{U}(1)$);
- one when E has CM not defined over K (matrices in $N(\mathrm{U}(1))$);
- one when E does not have CM (matrices in $\mathrm{SU}(2)$).

For illustrations, see https://math.mit.edu/~drew.

*The CM cases hold by results of Hecke. The non-CM case is the **Sato-Tate conjecture** and is known when K is totally real or a CM field, by work of many authors.

The Sato-Tate group of an abelian variety

Assume the **Mumford-Tate conjecture**† for A. Then there is a natural (but elaborate) construction of a compact Lie group $\mathrm{ST}(A)$ contained in $\mathrm{USp}(2g)$ and, for each $\mathfrak{p}$, a conjugacy class $\mathrm{Frob}_{\mathfrak{p}}$ in $\mathrm{ST}(A)$ with charpoly $\overline{L}_{\mathfrak{p}}(T)$. One conjectures (after Serre) that the $\mathrm{Frob}_{\mathfrak{p}}$ are equidistributed with respect to (the image of) Haar measure.

This reduces to a statement about analytic continuation of the L-functions associated to irreducible representations of $\mathrm{ST}(A)$. Besides CM cases, this is only known when it can be deduced via potential automorphy of Galois representations (as for elliptic curves over totally real or CM fields).

For $\dim(A) \leq 3$, $\mathrm{ST}(A)$ can be computed from the data of the $\mathbb{R}$-algebra $\mathrm{End}(A_{\overline{\mathbb{Q}}})_{\mathbb{R}} := \mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes_{\mathbb{Z}} \mathbb{R}$ and its $G_{\mathbb{Q}}$-action. This data can in principle be computed rigorously (Costa–Mascot–Sijsling–Voight).

†For any prime ℓ, the image of the ℓ-adic Galois representation of A has finite index in the maximal group allowed by the Hodge structure. This holds for $\dim(A) \leq 3$.

The connected and finite parts of the Sato-Tate group

There is a canonical exact sequence

$$1 \to \mathrm{ST}(A)^\circ \to \mathrm{ST}(A) \to \pi_0(\mathrm{ST}(A)) \to 1$$

where $\mathrm{ST}(A)^\circ$ is the identity component (and hence connected) and $\pi_0(\mathrm{ST}(A))$ is the component group (and hence finite).

The group $\mathrm{ST}(A)^\circ$ depends only on $A_{\overline{\mathbb{Q}}}$. It is equivalent data to the Mumford-Tate group (determined by the Hodge structure).

The group $\pi_0(\mathrm{ST}(A))$ is the Galois group of a certain finite extension L/K. For $\dim(A) \leq 3$, L is the **endomorphism field** of A: the minimal extension for which $\mathrm{End}(A_L) = \mathrm{End}(A_{\overline{\mathbb{Q}}})$.

For example, if $\dim(A) = 1$ and A has CM by a quadratic field M not in K, then $L = MK$ and $\mathrm{ST}(A)/\mathrm{ST}(A)^\circ = N(\mathrm{U}(1))/\mathrm{U}(1) \cong \mathrm{Gal}(MK/K)$.

Aside: a motivic generalization

The conjectural equidistribution of Frobenius classes in $\mathrm{ST}(A)$ is a special case of a conjecture about an arbitrary motive[‡] formulated by Serre. The group $\mathrm{ST}(A)$ is derived from the **motivic Galois group**.

In the special case of a motive of weight 0 (Artin motive), the motivic Galois group is just the usual Galois group, and the conjecture specializes to the Chebotarev density theorem.

There are many classes of motives of weight > 1 for which classification of Sato-Tate groups is of current interest (e.g., K3 surfaces), but those are topics for another day.

[‡] I ignore here the differences between various motivic categories, as Serre did.

The case of surfaces§

Theorem (Fité–K–Rotger–Sutherland, 2012)

There are 52 *conjugacy classes of closed subgroups of* $\mathrm{USp}(4)$ *which occur as* $\mathrm{ST}(A)$ *for some abelian surface* A *over some number field* K.

- This includes 6 options for $\mathrm{ST}(A)^\circ$; see next slide.
- $\#\pi_0(\mathrm{ST}(A))$ divides $48 = 2^4 \times 3$ (and this value occurs).
- The 52 cases correspond to distinct distributions of $\overline{L}_\mathfrak{p}$.
- The theorem is quantified over all K. If we require $K = \mathbb{Q}$, then 34 cases occur. If we require K to be totally real, then 35 cases occur.
- There is a field K over which all 52 cases occur (Fité–Guitart).
- Nothing changes if we restrict to principally polarized abelian surfaces or even Jacobians. FKRS give explicit genus 2 curves in all cases.

§This grew from work of K–Sutherland presented at AGCT in 2007; the collaboration with Fité and Rotger was catalyzed by Serre.

Identity components vs. extensions: the case of surfaces

$\mathrm{End}(A_{\overline{\mathbb{Q}}})_{\mathbb{R}}$	$\mathrm{ST}(A)^\circ$	Extensions	Maximal
$\mathbb{R}$	$\mathrm{USp}(4)$	1	1
$\mathbb{R} \times \mathbb{R}$	$\mathrm{SU}(2) \times \mathrm{SU}(2)$	2	1
$\mathbb{C} \times \mathbb{R}$	$\mathrm{U}(1) \times \mathrm{SU}(2)$	2	1
$\mathbb{C} \times \mathbb{C}$	$\mathrm{U}(1) \times \mathrm{U}(1)$	5	2
$\mathrm{M}_2(\mathbb{R})$	$\mathrm{SU}(2)_2$	10	2
$\mathrm{M}_2(\mathbb{C})$	$\mathrm{U}(1)_2$	32	2
Total		52	9

Here $*_2$ denotes the diagonal embedding.

Warning: if A is geometrically simple, $\mathrm{ST}(A)^\circ$ can still be decomposable because it only depends on $\mathrm{End}(A_{\overline{\mathbb{Q}}})_{\mathbb{R}}$. For example, if A has CM by a quartic field K, then $\mathrm{End}(A_{\overline{\mathbb{Q}}})_{\mathbb{R}} \cong K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C} \times \mathbb{C}$.

The case of threefolds

Theorem (Fité–K–Sutherland, in progress)

There are 410 *conjugacy classes of closed subgroups of* $\mathrm{USp}(6)$ *which occur as* $\mathrm{ST}(A)$ *for some abelian threefold* A *over some number field* K*.*

- This includes 14 options for $\mathrm{ST}(A)^\circ$ (Moonen–Zarhin).
- $\#\pi_0(\mathrm{ST}(A))$ divides[¶] one of $192 = 2^6 \times 3$, $336 = 2^4 \times 3 \times 7$, $432 = 2^4 \times 3^3$ (and these values occur).
- The 410 cases correspond to only 409 distinct distributions of $\overline{L}_{\mathfrak{p}}$. The two cases that collide have distinct component groups.
- We do not know what happens if we restrict K.
- Nothing changes if we require a principal polarization, but we do not yet know what happens for Jacobians. More on this later.

[¶] This refines earlier estimates by Silverberg and Guralnick-K.

Identity components vs. extensions: the case of threefolds

$\mathrm{End}(A_{\overline{\mathbb{Q}}})_{\mathbb{R}}$	$\mathrm{ST}(A)^\circ$	Extensions	Maximal
$\mathbb{R}$	$\mathrm{USp}(6)$	1	1
$\mathbb{C}$	$\mathrm{U}(3)$	2	1
$\mathbb{R}\times\mathbb{R}$	$\mathrm{SU}(2)\times\mathrm{USp}(4)$	1	1
$\mathbb{C}\times\mathbb{R}$	$\mathrm{U}(1)\times\mathrm{USp}(4)$	2	1
$\mathbb{R}\times\mathbb{R}\times\mathbb{R}$	$\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$	4	1
$\mathbb{C}\times\mathbb{R}\times\mathbb{R}$	$\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$	5	1
$\mathbb{C}\times\mathbb{C}\times\mathbb{R}$	$\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{SU}(2)$	5	2
$\mathbb{C}\times\mathbb{C}\times\mathbb{C}$	$\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)$	13	3
$\mathbb{R}\times\mathrm{M}_2(\mathbb{R})$	$\mathrm{SU}(2)\times\mathrm{SU}(2)_2$	10	2
$\mathbb{R}\times\mathrm{M}_2(\mathbb{C})$	$\mathrm{SU}(2)\times\mathrm{U}(1)_2$	32	2
$\mathbb{C}\times\mathrm{M}_2(\mathbb{R})$	$\mathrm{U}(1)\times\mathrm{SU}(2)_2$	31	2
$\mathbb{C}\times\mathrm{M}_2(\mathbb{C})$	$\mathrm{U}(1)\times\mathrm{U}(1)_2$	122	2
$\mathrm{M}_3(\mathbb{R})$	$\mathrm{SU}(2)_3$	11	2
$\mathrm{M}_3(\mathbb{C})$	$\mathrm{U}(1)_3$	171	12
Total		410	33

An initial subdivision

For each candidate G° for $\mathrm{ST}(A)^\circ$, candidates for G correspond to conjugacy classes of finite subgroups of N/G° where N is the normalizer of G° in $\mathrm{USp}(6)$. We distinguish four subcases.

- **Indecomposable:** $G^\circ = \mathrm{USp}(6), \mathrm{U}(3)$. In these cases, the only options are $\mathrm{USp}(6), \mathrm{U}(3), N(\mathrm{U}(3))$.
- **Split product:** G° factors as a nontrivial product $G_1^\circ \times G_2^\circ$ with no shared factors between the two sides (i.e., $\mathrm{U}(1) \times * \times \mathrm{SU}(2)$ or $* \times *_2$). In these cases, N splits as $N_1 \times N_2$, so we can reduce to the classification for elliptic curves and abelian surfaces.
- **Triple products**: $G^\circ = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$. In these cases, N/G° is finite.
- **Triple diagonals**: $G^\circ = \mathrm{SU}(2)_3, \mathrm{U}(1)_3$. In these cases, N/G° is infinite, but there is a bound on the order of elements in N/G° coming from the **rationality condition** (see below).

The upper bound: a group-theoretic classification

For each candidate G° for $\mathrm{ST}(A)^\circ$, we identify all extensions of G° within $\mathrm{USp}(6)$ satisfying the **rationality condition**: for every representation of $\mathrm{USp}(6)$, the average trace on each coset of G° is in $\mathbb{Z}$.

This gives the correct upper bound except when G° includes multiple factors of $\mathrm{U}(1)$, in which case one must rule out some cases using Shimura's theory of CM types. (For $G^\circ = \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$, $[N : G^\circ] = 48$ but $[G : G^\circ] \leq 8$.)

Most of the work occurs when $G^\circ = \mathrm{U}(1)_3$; in this case $N = \mathrm{U}(3) \rtimes \mathrm{C}_2$. The relevant subgroups of $\mathrm{U}(3)/\,\mathrm{U}(1)_3$ are found using the Blichfeldt–Dickson–Miller classfication of finite subgroups of $\mathrm{PSU}(3)$. For each such subgroup, the C_2-extensions are described (painfully) in terms of the normalizer within $\mathrm{U}(3)/\,\mathrm{U}(1)_3$.

The lower bound: realization by PPAVs

By base extension, for each G° it suffices to realize each *maximal* candidate for G using some principally polarized abelian threefold over $\mathbb{Q}$.

- For G° indecomposable, use generic hyperelliptic and Picard curves.
- For G° a split product, use products of lower-dimensional examples. In all cases except $G^\circ = \mathrm{U}(1) \times \mathrm{U}(1)_2$, we also find explicit examples of genus 3 curves.
- For $G^\circ = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1), \mathrm{SU}(2)_3$, we find explicit examples of genus 3 curves.
- For $G^\circ = \mathrm{U}(1)_3$, we realize G by twisting the cube of an elliptic curve with CM by an imaginary quadratic field M (then making an isogeny to get a principal polarization). The twist uses a Galois cycle valued in a subgroup[‖] of $\mathrm{GL}(3, \mathfrak{o}_M)$ with projective image G/G°.

[‖] These are almost all complex reflection groups, which makes it easy to solve the embedding problem needed to construct the cocycle.

Moment (and other) statistics

For G a closed subgroup of USp(6) and e_1, e_2, e_3 nonnegative integers, the **moment** M_{e_1,e_2,e_3} of G can be interpreted either as:

- the expected value of $a_1^{e_1} a_2^{e_2} a_3^{e_3}$ where $1 + a_1 T + \cdots + T^6$ is the charpoly of a random element of G;
- the dimension of the G-fixed subspace of $(\mathbb{C}^6)^{\otimes e_1} \otimes (\wedge^2 \mathbb{C}^6)^{\otimes e_2} \otimes (\wedge^3 \mathbb{C}^6)^{\otimes e_3}$. (This is a nonnegative integer!)

For our 410 groups, we obtain 409 distinct collections** of moments. The collision comes from two cases with identity component $\mathrm{U}(1)_3$ whose π_0's are distinct groups of order 54 with a common index-2 subgroup.

When comparing to L-function data, it is useful to also record the density of points on which a_1, a_2, a_3 are constant; e.g., for a non-CM elliptic curve, $a_1 = 0$ with density $1/2$. (By parity, only the value 0 can occur for a_1, a_3 with positive density, but a_2 can take other integer values.)

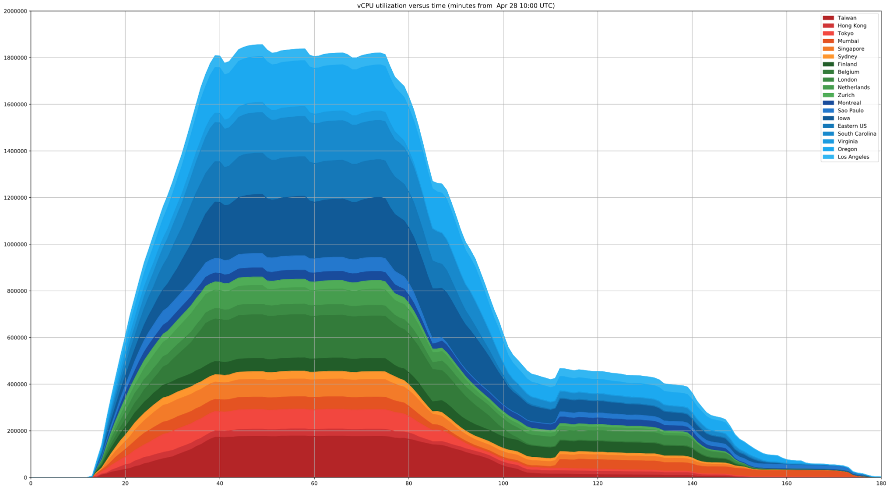
**It suffices to consider triples with $e_1 + e_2 + e_3 \leq 6$.

Into the cloud

We have made several extensive tabulations of genus 3 curves over $\mathbb{Q}$ to look for exotic Sato-Tate groups (and to test the completeness of the classification). I describe one of these here. We have also done specialized searches for hyperelliptic and Picard curves, and are working on other families with automorphisms (e.g., see Lercier–Ritzenthaler–Rovetta-Sijsling, Lorenzo García).

We used Google Cloud Platform to process 10^{17} quartic polynomials, looking for smooth curves with discriminant $< 10^9$ or divisible only by $2, 3, 5, 7$. This took 3 hours of wall time, using up to 1.8×10^6 vCPUs, and yielded 3.3×10^8 polynomials (representing 3.6×10^6 distinct isomorphism classes). This data is of independent interest, and should find its way into the LMFDB someday (compare Sutherland, ANTS 2018).

Visualization of a Google Cloud Platform run



Heuristic calculation of endomorphism algebras

For each of the 3.6×10^6 curves from GCP, we compute $L_p(T)$ for $p < 500$ (by point counting) and check whether for one of $\ell = 2, 3, 5$, the $L_p(T)$'s *cannot* be matched with a subset of a maximal subgroup of $\mathrm{GSp}(6, \mathbb{F}_\ell)$; if so, the Sato-Tate group must be USp(6). Setting such cases aside yields 6×10^5 curves requiring further analysis. (Aside: a handful of these also have Sato-Tate group USp(6); these may also be of interest!)

We then perform a heuristic calculation of the endomorphism algebra using the method (and code) of Costa–Mascot–Sijsling–Voight; this uses code of Neurohr to compute period lattices over $\mathbb{C}$ to high precision.

We plan to add to this (once suitable code is available):

- rigorous computation of endomorphism algebras;
- computation of $L_p(T)$ for $p \leq 2^{20}$ (say) to match moments and densities. (For hyperelliptic curves, use Sutherland's `smalljac`.)

Gotta catch 'em all!

Of the 410 possible Sato–Tate groups for abelian threefolds, the 33 maximal ones (for inclusions of finite index) occur for principally polarized abelian threefolds over $\mathbb{Q}$. Can we find explicit curves of genus 3 over $\mathbb{Q}$ for all 33 cases? If so, it would follow that every possible Sato-Tate group of an abelian threefold occurs for some genus 3 curve over some K.

Theorem (Fité–K–Sutherland, in progress)

For G a maximal Sato–Tate group for abelian threefolds with $G^\circ \neq \mathrm{U}(1) \times \mathrm{U}(1)_2, \mathrm{U}(1)_3$, we have an explicit curve C of genus 3 over $\mathbb{Q}$ with $\mathrm{ST}(\mathrm{Jac}(C)) \cong G$.

For $G^\circ \cong \mathrm{U}(1)_3$, we have explicit examples in 3 of the 12 cases with $G^\circ \cong \mathrm{U}(1)_3$, and existence arguments in a few more cases.

We describe here some techniques that can be used to identify suitable examples and/or target our cloud searches.

Numerical reconstruction from periods

One can construct abelian varieties with exotic endomorphism algebras by making high-precision computations with their period lattices and polarizations, as in the heuristic computation of endomorphism algebras.

This has been done successfully in numerous cases, including:

- CM curves of genus 2 (van Wamelen);
- CM Picard curves of genus 3 (Koike–Weng);
- RM curves of genus 2 (Kumar–Mukamel);
- CM hyperelliptic curves of genus 3 (Balakrishnan–Ionica–Lauter–Vincent);
- CM plane quartic curves (Kılıçer–Labrande–Lercier–Ritzenthaler–Sijsling–Streng).

The upcoming thesis of Hanselman (under Sijsling) is in this direction.

Automorphism groups

Most exotic examples have nontrivial (nonhyperelliptic) automorphisms. There are several reasons for this.

- Since $\pi_0(\mathrm{ST}(A))$ is the Galois group of the endomorphism field, automorphisms not defined over the ground field contribute directly.
- Automorphisms also tend to force the Jacobian to be decomposable. For example, these give the best examples to date of high-genus Jacobians with many elliptic factors (Ekedahl–Serre, Paulhus).
- One can twist a curve using automorphisms and then control the resulting Sato-Tate group. This was done for the Fermat/Klein quartics by Fité–Lorenzo García–Sutherland and for

$$C: y^2 = x^8 - 14x^4 + 1 \qquad (\#\mathrm{Aut}(C) = 48)$$

by Arora–Cantoral–Landesman–Lombardo–Morrow; these give some extensions of $\mathrm{U}(1)_3$ and $\mathrm{SU}(2)_3$, respectively.

Action of C_2

The generic hyperelliptic and nonhyperelliptic curves of genus 3 with an extra involution are

$$y^2 = P(x^2), \qquad Y^4 + P_2(X,Z)Y^2 + P_4(X,Z) = 0.$$

In the hyperelliptic case, the quotient is $y^2 = P(x)$ and the Prym is the Jacobian of $y^2 = xP(x)$ (which is also a quotient).

In the nonhyperelliptic case, the quotient is

$$y^2 + P_2(x,1)y + P_4(x,1) = 0;$$

the Prym is identified by Ritzenthaler–Romagny using a result of Bruin (when $P_4(x,1)$ factors into two quadratics, else some descent is needed).

This gives many examples of split/triple products, but not $\mathrm{U}(1) \times \mathrm{U}(1)_2$.

Actions of $\mathrm{C}_2 \times \mathrm{C}_2$

The generic nonhyperelliptic curve with an action of $\mathrm{C}_2 \times \mathrm{C}_2$ is

$$aX^4 + bY^4 + cZ^4 + dX^2Y^2 + eX^2Z^2 + fY^2Z^2 = 0.$$

Its Jacobian is isogenous to the products of the Jacobians of the quotients by the three involutions. We obtain several examples by forcing some of these quotients to be Galois conjugate.

One can also work backwards. Given three elliptic curves E_1, E_2, E_3 with "compatible" 2-torsion, Howe–Leprévost–Poonen produce a curve of the above form with (twists of) E_1, E_2, E_3 as the Jacobians of the quotients.

Glueing elliptic curves: an example of Everett Howe

Consider the following elliptic curves over $\mathbb{Q}$.

$$\begin{aligned} E_1 : y^2 &= x^3 + 3x^2 + 3x \quad \text{CM by } \mathbb{Q}(\zeta_3) \\ E_2 : y^2 &= x^3 + x^2 + 2x \quad \text{CM by } \mathbb{Q}(\sqrt{-2}) \\ E_3 : y^2 &= x^3 - 21x \quad \text{CM by } \mathbb{Q}(i) \end{aligned}$$

Then $E_1 \times E_2 \times E_3$ is isogenous to a twist of the Jacobian of

$$3X^4 + 2Y^4 + 6Z^4 - 6X^2Y^2 + 6X^2Z^2 - 12Y^2Z^2 = 0.$$

This realizes a maximal extension of $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$ with component group $\mathrm{C}_2 \times \mathrm{C}_2 \times \mathrm{C}_2$.

By varying the curves, we can obtain 16 examples of this type. Are these (up to twists) the only curves over $\mathbb{Q}$ with this Sato-Tate group?

Potential examples from twisting abelian threefolds

Most of the missing examples are for extensions of $\mathrm{U}(1)_3$. These are all known to occur for twists of the cube of a CM elliptic curve over $\mathbb{Q}$. In some cases, we can show that there exists an isogeny to an abelian threefold with an indecomposable principal polarization; up to twist, this is the Jacobian of some curve. Can one compute this curve?

For example, let E be an elliptic curve over $\mathbb{Q}$ with CM by $\mathbb{Q}(\zeta_3)$. Then there exist a twist A of E^3 and a 3-isogeny $A \to B$ such that B is (indecomposably) principally polarized and $\pi_0(\mathrm{ST}(A)) \cong \pi_0(\mathrm{ST}(B))$ is a double cover of the Hessian group of order 216 (the symmetries of the configuration of flexes of a plane cubic).

For this and other examples, even partial information about the shape of the curve would be helpful for executing brute force searches.

Thank you for your attention!

... positive characteristic?

One can ask (and presumably answer) similar questions where K is a function field over a finite field. However, our results are not directly applicable, because they depend on constraints from Hodge theory which do not apply in positive characteristic. For instance, $\mathrm{ST}(A)^\circ$ need not be positive-dimensional because of isotrivial abelian varieties.

... abelian fourfolds?

There are many reasons to be wary.

- The number of cases should grow into the thousands. More of the process will need to be automated, particularly finding finite subgroups of $N(\mathrm{U}(1)_4)/\mathrm{U}(1)_4$ satisfying the rationality condition.
- By examples of Mumford and Shioda, the Sato-Tate group can be smaller than what is predicted by the endomorphism algebra (and in such cases the Mumford-Tate conjecture is also at issue).
- On a related note, the real endomorphism algebra can now include $\mathbb{H}$.
- The rationality condition is probably too weak, due to the distinction between fields of definition and fields of traces for linear representations.
- The analysis of CM types is more involved than before.

... other motives?

It might be more productive to focus on other classes of motives.

For K3 surfaces, cases of high Picard number (≥ 17) are handled by the classification for abelian surfaces. Additional cases can probably be handled using the classification of finite subgroups of $\mathrm{SO}(n, \mathbb{Z})$; this should be feasible (and possibly even known) with current technology.

For Calabi–Yau threefolds, with Fité and Sutherland we treated the case where the primitive middle cohomology has Hodge numbers $(1, 1, 1, 1)$ (e.g., in the Dwork pencil).