Witt-perfect rings and almost purity

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Witt vectors in arithmetic, geometry, and topology
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http://www.math.uci.edu/~davis/Frobenius.pdf.

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Preface: some related talks

This talk is something of a hybrid of two previous talks:

- Absolute de Rham cohomology? A fantasy in the key of p (Nagoya 2010):
  http://math.ucsd.edu/~kedlaya/nagoya2010.pdf
- Towards uniformity over p in p-adic Hodge theory (Lyon 2011):
  http://math.ucsd.edu/~kedlaya/lyon2011.pdf

See the former for some broad context, and the latter for some specific manipulations in p-adic Hodge theory related to those described here.

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Witt vectors and their Frobenius maps

Throughout this talk, $\mathbb{N}$ is the multiplicative monoid $\{1, 2, \dots\}$, and all rings are commutative and unital.

Theorem (Witt)

There exists a unique functor $W : \mathbf{Ring} \to \mathbf{Ring}$ whose underlying functor $\mathbf{Ring} \to \mathbf{Set}$ is the functor $\bullet^{\mathbb{N}}$ and for which the ghost maps

$$w : W(R) \to R^{\mathbb{N}}, \qquad (x_n)_{n \in \mathbb{N}} \mapsto (w_n)_{n \in \mathbb{N}}, \qquad w_n := \sum_{d|n} d x_d^{n/d}$$

define a natural transformation $W \to \bullet^{\mathbb{N}}$ of functors of rings. Moreover, for $m \in \mathbb{N}$, there is a unique natural transformation $F_m : W \to W$ with

$$D_m \circ w = w \circ F_m$$

for $D_m : \bullet^{\mathbb{N}} \to \bullet^{\mathbb{N}}$ the decimation map $(x_n)_{n \in \mathbb{N}} \mapsto (x_{mn})_{n \in \mathbb{N}}$.

Truncation sets

A *truncation set* is a subset $S \subseteq \mathbb{N}$ closed under taking divisors. Projecting onto the components indexed by S defines a new functor

$$W_S : \mathbf{Ring} \to \mathbf{Ring}.$$

For $m \in \mathbb{N}$, the map $F_m : W \to W$ induces a map $F_m : W_S \to W_{S/m}$ for

$$S/m := \{n \in \mathbb{N} : mn \in S\}.$$

For $m \in \mathbb{N}$, we will write W_n to mean W_S where $S = \{d \in \mathbb{N} : d|n\}$.

E.g., if $S = \{1, p, \dots\}$, this gives the *infinite p-typical Witt vectors* W_{p^∞}. If $S = \{1, p, \dots, p^n\}$, we get *finite p-typical Witt vectors* W_{p^n}. Note that F_p maps $W_{p^{n+1}}$ to W_{p^n}. (When $p = 0$ in R, F_p can be obtained by applying the p-power map to each component and forgetting the last one; but in general it does not lift naturally to a map $W_{p^{n+1}} \to W_{p^{n+1}}$.)

Witt-perfect rings

For the moment, fix a prime number p. We say a ring R is *Witt-perfect (at p)* if for all nonnegative integers n, the map

$$F_p : W_{p^{n+1}}(R) \to W_{p^n}(R)$$

is surjective.

Warning: this is strictly weaker than saying that

$$F_p : W_{p^\infty}(R) \to W_{p^\infty}(R)$$

is surjective! More on this later.

Remark: the kernel of F_p can be described well. The first components of elements of the kernel form a certain explicit ideal. If R is p-torsion-free, every element of the kernel is determined by its first component (because the ghost map is injective).

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Witt-perfectness for big Witt vectors

One can also state the Witt-perfectness condition using big Witt vectors. This will become relevant later when we want to impose the condition at more than one prime at once.

Theorem (not so difficult)

For p a prime and R a ring, the following conditions are equivalent.

(a) *The ring R is Witt-perfect at p.*

(b) *For every finite truncation set S,*

$$F_p : W_S(R) \to W_{S/p}(R)$$

is surjective. (Note that S need not contain only powers of p.)

Question: Is there an analogous statement for Dress-Siebeneicher rings?

Contents

Valuation rings

If R is a valuation ring of mixed characteristic $(0, p)$, then R is Witt-perfect if and only if the p-power map on R/pR is surjective and R is *not discrete*.

For example, the valuation ring of $\mathbb{Q}_p(\mu_{p^\infty})$ is $\mathbb{Z}_p[\mu_{p^\infty}]$, which is evidently Witt-perfect. Note that it is not necessary to take the p-adic completion!

In addition, the valuation ring of any algebraic extension of $\mathbb{Q}_p(\mu_{p^\infty})$ is Witt-perfect. This is far less evident! It can be deduced using the Fontaine-Wintenberger theorem, or later using the almost purity theorem.

By contrast, $\mathbb{Z}_p$ is not Witt-perfect. More on this shortly.

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A finite criterion for Witt-perfectness

If R is Witt-perfect, then $F_p : W_p(R) \to W_1(R)$ is surjective, i.e., $(x_1, x_p) \to x_1^p + px_p$ is surjective. So p-th powering on R/pR is surjective.

The converse is true if $p = 0$ in R but otherwise false. E.g., for $R = \mathbb{Z}_p$, the p-th power map on R/pR is surjective but $F_p : W_{p^2}(R) \to W_p(R)$ is not: its image does not contain $V_p(1)$.

In some sense, this example is typical!

Theorem

A ring R is Witt-perfect if and only if $F_p : W_{p^2}(R) \to W_p(R)$ is surjective.

This means Witt-perfectness of R depends only on R/p^2R. E.g., since $\mathbb{Z}_p[\mu_{p^\infty}]$ is Witt-perfect, so then is $\mathbb{Z}[\mu_{p^\infty}]$.

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Surjectivity of the infinite Frobenius

If R is Witt-perfect and not of characteristic p, the map

$$F_p : W_{p^\infty}(R) \to W_{p^\infty}(R)$$

need not be surjective, e.g., for $R = \mathbb{Z}[\mu_{p^\infty}]$ or $R = \mathcal{O}_{\mathbb{C}_p}$.

The obstruction to this stronger surjectivity can be phrased as a *spherical completeness* condition. For instance, if R is a valuation ring of mixed characteristic which is Witt-perfect, then $F_p : W_{p^\infty}(R) \to W_{p^\infty}(R)$ is surjective if and only if certain decreasing sequences of balls with positive limiting radius are guaranteed to have nonempty intersections.

One may infer from this that surjectivity of the infinite Frobenius is probably not so important for applications.

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More examples

Theorem

A ring R is Witt-perfect if and only if $F_p : W_p(R) \to W_1(R)$ is surjective and the image of $F_p : W_{p^2}(R) \to W_p(R)$ contains $(0,1)$.

Consequently, if R is an algebra over a Witt-perfect ring, then R is Witt-perfect if and only if the p-power map on R/pR is surjective. This yields some additional examples of Witt-perfect rings:

$$\bigcup_{m=1}^{\infty} \mathbb{Z}[\mu_{p^\infty}, T_1^{p^{-m}}, \dots, T_n^{p^{-m}}], \quad \bigcup_{m=1}^{\infty} \mathbb{Z}[\mu_{p^\infty}, T_1^{\pm p^{-m}}, \dots, T_n^{\pm p^{-m}}].$$

More generally, if R is Witt-perfect and M is a toric monoid, then

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Almost purity, part 1

In this section, continue to fix a prime p. Let R be a p-torsion-free, Witt-perfect ring such that R is integrally closed in $R_p := R[p^{-1}]$.

Theorem (Almost purity, part 1)

Let S be the integral closure of R in a finite étale extension of R_p. Then S is also Witt-perfect.

For example, since $R = \mathbb{Z}[\mu_{p^\infty}]$ is Witt-perfect, so is the integral closure of R (or equivalently of $\mathbb{Z}$) in any finite extension of $\mathrm{Frac}(R)$. This alone is crucial for p-adic Hodge theory.

However, one can make some more precise statements by adopting Faltings' language of *almost ring theory*. We will introduce just enough of this language to say what we want, but it can be developed far further. (Usually this is only done assuming R contains a valuation ring; see for example the book of Gabber-Ramero.)

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A bit of almost ring theory

By a *p-ideal* of R, we will mean an ideal I for which $I^m \subseteq pR$ for some positive integer m. An R-module M is *almost zero* if it is killed by every p-ideal; such modules form a Serre subcategory. We use *almost* an an adjective/adverb to indicate properties of objects in the quotient category, e.g., an *almost isomorphism*.

An R-module M is *almost finite projective* if for every p-ideal I, there exist a finite free R-module F and R-module maps $M \to F \to M$ whose composition is multiplication by t for some $t \in R$ with $I \subseteq tR$. In particular, $M_p := M[p^{-1}]$ is a finite projective R_p-module.

An R-algebra S is *almost finite étale* if S is an almost finite projective R-module and the images of S (via the trace pairing) and $\mathrm{Hom}_R(S, R)$ in $\mathrm{Hom}_{R_p}(S_p, R_p)$ are almost equal.

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Almost purity, part 2

Theorem (Almost purity, part 2)

Let S_0 be the integral closure of R in a finite étale extension of R_p. Let S be an R-subalgebra of S_0 which is almost equal to S_0. Then S is an almost finite étale R-algebra.

Sketch of proof.

There is a unique absolute value $|\cdot|_p$ on R_p and S_p such that for $m \in \mathbb{Z}$ and $x \in S_p$, $|x| < p^{-m}$ if and only if $(p^{-m}x)$ is a p-ideal in S. Note that R and S are almost equal (but possibly not equal!) to the valuation subrings of R_p and S_p. With a quick argument, one reduces to the case where R is the valuation ring of a uniform $\mathbb{Q}_p$-Banach algebra.

This case is due to Kedlaya-Liu and Scholze: localize in nonarchimedean analytic geometry to reduce to the case of a valuation ring in a complete algebraically closed field, which is easy. □

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Warning: rampant speculation ahead!

This section of the talk is rated R for rampant speculation.

Overconvergent Witt vectors

Define $R, S, |\cdot|_p$ as in the previous section. Then extend $|\cdot|_p$ to $W_{p^n}(R)$:

$$|(x_1, x_p, \dots, x_{p^n})|_p = \max\{|x_{p^i}|_p^{p^{-i}}\}.$$

For $r > 0$, let $\underleftarrow{W}^r(\bullet_p)$ be the subring of the inverse limit

$$\cdots \stackrel{F_p}{\to} W_p(\bullet_p) \stackrel{F_p}{\to} W_1(\bullet_p)$$

consisting of coherent sequences $(\cdots, \underline{x}^{(p)}, \underline{x}^{(1)})$ such that $\sup_n\{r^{-n}|\underline{x}^{(p^n)}|_p^{p^n}\} < +\infty$.

Theorem (in progress!)

For each $r > 0$, $\underleftarrow{W}^r(S_p)$ is finite étale over $\underleftarrow{W}^r(R_p)$.

In the next few slides, we'll compare this with an existing construction in positive characteristic, which appears prominently in p-adic Hodge theory.

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Comparison with positive characteristic, part 1

Let E^+ denote the inverse limit

$$\cdots \stackrel{\varphi}{\to} R/pR \stackrel{\varphi}{\to} R/pR,$$

where φ is the p-power Frobenius; this ring is perfect of characteristic p.

There is a natural map

$$\varprojlim_{F_p} W_{p^\infty}(R) \to \varprojlim_{F_p} W_{p^\infty}(E^+) \cong W_{p^\infty}(E^+).$$

If R is the integral subring of a $\mathbb{Q}_p$-Banach algebra, then this map is a bijection. Moreover, if we project onto $W_1(R)$ to obtain a surjective homomorphism

$$\theta : W_{p^\infty}(E^+) \to R,$$

then the ideal $\ker(\theta)$ is principal. (This is *almost* still true if R is p-adically separated and complete.)

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Comparison with positive characteristic, part 2

Define the absolute value $|\cdot|_E$ on E^+ by the formula

$$|(\dots, r_p, r_1)|_E = \lim_{n\to\infty} |r_{p^n}|^{p^n}.$$

Obtain E from E^+ by inverting some element $x = (\dots, x_p, x_1)$ such that $x_p \in R/pR$ lifts to $\tilde{x}_p \in R$ with $\tilde{x}_p^p \equiv p \pmod{p^2R}$. One may then extend $|\cdot|_E$ to E so that $|xy|_E = |x|_E|y|_E$ for all $y \in E^+$.

For $r > 0$, let $\underleftarrow{W}^r(E)$ be the subring of $W_{p^\infty}(E)$ consisting of $(x_1, x_p, \dots)$ for which $\sup_n\{r^{-n}|x_{p^n}|^{p^{-n}}\} < +\infty$.

Theorem

For $r > 0$, there is a natural map $\underleftarrow{W}^r(R_p) \to \underleftarrow{W}^r(E)$, which is an isomorphism if R is p-adically separated and complete.

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Obtain E from E^+ by inverting some element $x = (\dots, x_p, x_1)$ such that $x_p \in R/pR$ lifts to $\tilde{x}_p \in R$ with $\tilde{x}_p^p \equiv p \pmod{p^2 R}$. One may then extend $|\cdot|_E$ to E so that $|xy|_E = |x|_E |y|_E$ for all $y \in E^+$.

For $r > 0$, let $\underleftarrow{W}^r(E)$ be the subring of $W_{p^\infty}(E)$ consisting of $(x_1, x_p, \dots)$ for which $\sup_n \{r^{-n} |x_{p^n}|^{p^{-n}}\} < +\infty$.

Theorem

For $r > 0$, there is a natural map $\underleftarrow{W}^r(R_p) \to \underleftarrow{W}^r(E)$, which is an isomorphism if R is p-adically separated and complete.

A case from p-adic Hodge theory

Let R be the p-adic completion of $\mathbb{Z}[\mu_{p^\infty}]$. Then

$$\mathcal{O}_{\tilde{\mathcal{E}}^\dagger} := \bigcup_{r>0} \underleftarrow{W}^r(E) \cong \bigcup_{r>0} \underleftarrow{W}^r(R)$$

is one of Fontaine's big rings (the *overconvergent perfect integral Robba ring*). It has a natural action of $\Gamma := \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ and of the bijective Frobenius map φ.

There is then a natural correspondence between continuous actions of $G_{\mathbb{Q}_p}$ on finite free $\mathbb{Z}_p$-modules and finite free $(\cup_{r>0} \underleftarrow{W}^r(E))$-modules with suitable actions of φ and Γ (called (φ, Γ)-*modules*).

Various results in p-adic Hodge theory can be interpreted as saying that one can embed the whole theory of motives over $\mathbb{Q}_p$ into a certain theory of (φ, Γ)-modules. Something like this is even true whenever R is p-adically complete, as in the work of K-Liu and Scholze.

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A variation

Take $R = \mathbb{Z}_p[\mu_{p^\infty}]$. We get a natural injective *but not surjective* map

$$\bigcup_{r>0} \underleftarrow{W}^r(R_p) \hookrightarrow \mathcal{O}_{\tilde{\mathcal{E}}^\dagger}$$

Both sides carry an action of Γ, but the source is not stable under φ.

However, $\bigcup_{r>0} \underleftarrow{W}^r(R_p)$ is stable under φ^{-1}! That is because φ^{-1} can be computed as the right shift:

$$(\cdots, \underline{x}^{(p)}, \underline{x}^{(1)}) \mapsto (\cdots, \underline{x}^{(p^2)}, \underline{x}^{(p)}).$$

Note that this implicitly uses the restriction maps $W_{p^{n+1}} \to W_{p^n}$.

These statements are also true if we take $R = \mathbb{Z}[\mu_{p^\infty}]$.

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Descent of (φ, Γ)-modules

A theorem of Cherbonnier-Colmez asserts that all étale (φ, Γ)-modules descend uniquely to the intersection of $\bigcup_{r>0} \underleftarrow{W}^r(\mathbb{Q}_p(\mu_{p^\infty}))$ with

$$\bigcup_{m \geq 0} \varprojlim \left(\cdots \stackrel{F_p}{\to} W_p(\mathbb{Q}_p(\mu_{p^{m+1}})) \stackrel{F_p}{\to} W_1(\mathbb{Q}_p(\mu_{p^m})) \right).$$

However, not all étale (φ, Γ)-modules descend to $\bigcup_{r>0} \underleftarrow{W}^r(\mathbb{Q}(\mu_{p^\infty}))$. One obstruction: by projection, one gets a semilinear action of Γ on a finite-dimensional $\mathbb{Q}_p(\mu_{p^\infty})$-vector space. Some open subgroup must act linearly, but the characteristic polynomials need not have coefficients in $\mathbb{Q}(\mu_{p^\infty})$.

Question: Do de Rham (φ, Γ)-modules descend to $\bigcup_{r>0} \underleftarrow{W}^r(\mathbb{Q}(\mu_{p^\infty}))$? How about Hodge-Tate modules, or modules with integral Sen weights?

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... and beyond: a version with big Witt vectors, part 1

Let R be a $\mathbb{Z}$-torsion-free ring which is Witt-perfect at every prime and integrally closed in $R_{\mathbb{Q}} := R \otimes_{\mathbb{Z}} \mathbb{Q}$. Let S be the integral closure of R in a finite étale extension of $R_{\mathbb{Q}}$; note that there exists some positive integer N for which $S[N^{-1}]$ is finite étale over $R[N^{-1}]$. By almost purity, S is again Witt-perfect at every prime.

Guess 1: Using N-ideals instead of p-ideals to define *almost*, S is an almost finite étale R-algebra.

Guess 2: Using a suitable definition (which is far from clear!), one can define $\underleftarrow{W}^r(S_{\mathbb{Q}})$ and have it be finite étale over $\underleftarrow{W}^r(R_{\mathbb{Q}})$. These rings will then contain formal inverses of F_n for all $n \in \mathbb{N}$.

Guess 3: One can realize étale (φ, Γ)-modules using $\underleftarrow{W}^r(\mathbb{Q}_p[\mu_{p^\infty}])$ in this sense. These would be finite projective modules with actions of Γ and the monoid of formal inverses of the F_n. (A similar guess was already made in my Lyon talk.)

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Guess 4: One can construct a functor from motives over $\mathbb{Q}$ to étale (φ, Γ)-modules over $\underleftarrow{W}^r(\mathbb{Q}[\mu_{p^\infty}])$ in this sense. For Artin motives, this should follow easily from Guess 2; in general, one should first work out relative comparison theory in p-adic Hodge theory using the methods of K-Liu and/or Scholze.

Guess 5: One can meaningfully add archimedean places to this discussion. Doing p-adic Hodge theory at the infinite place should result in something resembling ordinary Hodge theory, but possibly with a more noncommutative flavor.

Guess 6: For $R = \mathbb{Z}[\mu_\infty]$, this all has some arithmetic meaning...? One might hope in particular for links to Bost-Connes systems, and ultimately to complex and p-adic L-functions.

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