# Almost purity and overconvergent Witt vectors

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# Context: absolute de Rham cohomology

This talk is part of an ongoing project to investigate the properties of a hypothetical analogue of crystalline cohomology (a $p$-adic Weil cohomology theory for smooth proper varieties in characteristic $p$) related to $L$-functions of varieties over $\mathbb{Q}$. Additional references:

- Absolute de Rham cohomology? A fantasy in the key of $p$ (slides, Nagoya 2010): `http://kskedlaya.org/nagoya2010.pdf`.
- Towards uniformity over $p$ in $p$-adic Hodge theory (slides, Lyon 2011): `http://kskedlaya.org/lyon2011.pdf`.
- Rational structures and $(\varphi, \Gamma)$-modules (preprint): `http://kskedlaya.org/papers/`.
- On the Witt vector Frobenius (to appear in *Proc. AMS*).

# Witt vectors

Throughout this talk, fix a prime $p$.

Let $W_{p^n}$ denote the functor of $p$-typical Witt vectors of length $n+1$. For instance, $W_{p^n}(\mathbb{F}_p) \cong \mathbb{Z}_p/(p^{n+1})$. More generally, if $R$ is a perfect ring of characteristic $p$, then $W_{p^n}(R)$ is the truncation mod $p^{n+1}$ of the unique $p$-adically complete ring $S$ with $S/(p) \cong R$.

However, $W_{p^n}$ is a meaningful functor on arbitrary rings! It is characterized by the underlying functor on sets being the $(n+1)$-fold product and the fact that the *ghost maps*

$$w_{p^m}(x_1, \dots, x_{p^n}) = \sum_{i=0}^{m} p^i x_{p^i}^{p^{m-i}} \qquad (m = 0, \dots, n)$$

are ring homomorphisms. (Note: $w_1 = x_1$, $w_p = x_1^p + p x_p$, etc.)

We also write $W(R)$ for $\varprojlim_n W_{p^n}(R)$, the functor of $p$-typical vectors of infinite length. (The maps in the inverse limit are initial segments.)

# Witt-perfect rings

For each nonnegative integer $n$, there is a natural transformation $F : W_{p^{n+1}} \to W_{p^n}$ characterized by the property that

$$w_{p^m} \circ F = w_{p^{m+1}}.$$

This is called *Frobenius* because for $R$ of characteristic $p$, the map $F : W(R) \to W(R)$ is also the one induced by the $p$-power map on $R$.

The ring $R$ is *Witt-perfect* if the maps $F : W_{p^{n+1}}(R) \to W_{p^n}(R)$ are all surjective[1]. There are many[2] equivalent characterizations, e.g.: the $p$-power map on $R/(p)$ is surjective *and* there exists $x \in R$ with $x^p \equiv p \pmod{p^2}$.

[1]Warning: this generally does not imply that $F : W(R) \to W(R)$ is surjective.
[2]See "On the Witt vector Frobenius" for further discussion.

# Examples

Some typical examples of Witt-perfect rings are:

- the $p$-cyclotomic ring $\mathbb{Z}[\mu_{p^\infty}]$;
- the ring $\mathbb{Z}[\mu_{p^\infty}][T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}]$.

In some sense, this property is ubiquitous.

**Lemma**

*For any local ring $R$, the direct limit of the rings $R[U^{p^{-m}}]$, where $m$ runs over nonnegative integers and $U$ runs over finite subsets of $R^\times$, is Witt-perfect.*

**Corollary**

*All local rings in the syntomic (flat, local complete intersection, étale away from $p$) topology are Witt-perfect.*

# Almost purity, version 1

A ring $R$ is *p-normal* if it is $p$-torsion-free and integrally closed in $R_p = R[p^{-1}]$.

Theorem

*Let $R$ be a $p$-normal Witt-perfect ring. Let $S$ be the integral closure of $R$ in a finite étale extension of $R_p$. Then $S$ is again Witt-perfect (and is* **almost finite étale** *over $R$).*

This reduces to the case where $R$ is $p$-adically complete. In this case, $R$ is Witt-perfect if and only if it is *integral perfectoid* in the sense of Scholze and Kedlaya-Liu, so one can use techniques of analytic geometry (adic spaces) to further reduce to the case where $R_p$ is a complete field.

# Inverse limits along Frobenius

Let $\underleftarrow{W}(R)$ be the inverse limit of the rings $W_{p^m}(R)$ along Frobenius (rather than restriction). If $R$ is not Witt-perfect, this inverse limit can be small, e.g., $\underleftarrow{W}(\mathbb{Z}_p) \cong \mathbb{Z}_p$.

**Theorem**

*If $R$ is $p$-adically complete and Witt-perfect, then $\underleftarrow{W}(R) \cong W(\varprojlim_F R/pR)$.*

For example, if $R = \mathcal{O}_{\mathbb{C}_p}$, then $R$ is Witt-perfect and $\varprojlim_F R/pR$ is a valuation ring of a complete algebraically closed field of characteristic $p$. This ring appears in the construction of Fontaine's period rings.

Warning: $\underleftarrow{W}(R)$ admits a map $F^{-1}$, but not $F$ in general even if $R$ is Witt-perfect (unless $R$ is also $p$-adically complete).

# Norms on Witt vectors

Let $R$ be a $p$-normal ring. Then $R_p$ carries a natural *p-adic norm* $|\bullet|_p$.

Define a norm $|\bullet|_W$ on $W_{p^n}(R)$ and $W(R)$ by the formula

$$\left|(x_{p^i})\right|_W = \sup_i \{ \left|x_{p^i}\right|_p^{p^{-i}} \}.$$

**Theorem**

*This formula defines a submultiplicative[a] norm on $W_{p^n}(R)$ and a power-multiplicative[b] norm on $W(R)$.*

[a] In particular, $|xy|_W \leq |x|_W \, |y|_W$.
[b] In particular, $\left|x^2\right|_W = |x|_W^2$.

This may be viewed as an analogue of Gauss's lemma for polynomials.

# Overconvergent Witt vectors

Fix $b > 0$. Define the subset $\underleftarrow{W}^b(R)$ of $\underleftarrow{W}(R)$ consisting of those elements $(\underline{x}_{p^{-n}})_{n=0}^{\infty}$ such that $p^{-bn} \left|\underline{x}_{p^{-n}}\right|_W^{p^n} \to 0$ as $n \to \infty$.

**Theorem**

*The subset $\underleftarrow{W}^b(R)$ is a subring of $\underleftarrow{W}(R)$. Moreover, the formula*

$$\left|(\underline{x}_{p^{-n}})\right|_b = \sup_n \{p^{-bn} \left|\underline{x}_{p^{-n}}\right|_W^{p^n}\}$$

*defines a submultiplicative norm which is power-multiplicative for $b \leq 1$ and equivalent to a power-multiplicative norm for $b > 1$.*

# Almost purity, version 2

**Theorem**

*Let $R$ be a $p$-normal Witt-perfect ring. Let $S$ be the integral closure of $R$ in a finite étale extension of $R_p$. Then for all $b > 0$, $\underleftarrow{W}^b(S)$ is finite étale over $\underleftarrow{W}^b(R)$.*

When $R$ is $p$-adically complete and Witt-perfect and $b < 1$, this gives a well-studied ring in $p$-adic Hodge theory; for instance, such rings can be used to describe $p$-adic Galois representations via $(\varphi, \Gamma)$-*modules*. In particular, the statement of the theorem in this case is given explicitly by Kedlaya-Liu.

For general $R$, the extra work required to reduce to the complete case is mostly proving that the ring extension is finite.

# Where is this going?

Recall that our original motivation was to globalize crystalline cohomology. A more immediate goal is to revisit the comparison isomorphisms between de Rham, étale, and crystalline cohomology for varieties over $p$-adic fields, in order to incorporate the rational structure on de Rham cohomology (including the Hodge filtration) when the variety descends to a number field.

The plan is to construct the comparison isomorphism (via Wach modules in the good reduction case) using the de Rham-Witt complex. Overconvergent Witt vectors are likely to play a key role.