# Sato-Tate groups of motives


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# Philosophy: monodromy in the arithmetic direction

When one speaks of *monodromy* in either a topological or $\ell$-adic sense, one is usually considering a *geometric family* of algebraic varieties, i.e., the base space is itself a variety over a field.

But an algebraic variety over a number field can itself be viewed as a family by choosing an integral model. The fibers of this family are varieties over varying finite fields; a good analogue of the global monodromy group in this setting is a certain compact real Lie group called the *Sato-Tate group* of the family (or motive).

We will mostly only consider fibers of good reduction, for which there is no dependence on the choice of model. For bad reduction fibers, one should be careful about models; one then encounters *local monodromy* which helps control the global monodromy. But we'll ignore this here.

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# Example: the Chebotarev density theorem

Let $L/K$ be a Galois extension of number fields with Galois group $G$. For each prime $\mathfrak{p}$ of $K$ not dividing the discriminant of $L$, the Artin map associates to $\mathfrak{p}$ a conjugacy class $g_{\mathfrak{p}}$ in $G$. By the Chebotarev density theorem, these are equidistributed for the measure

$$\mu(C) = \frac{\#C}{\#G}.$$

That is, for any continuous function $f : \mathrm{Conj}(G) \to \mathbb{C}$, the average of $f(g_{\mathfrak{p}})$ over $\mathfrak{p}$ equals $\int_\mu f$.

The measure $\mu$ can be viewed as the image of the Haar measure of $G$, viewed as a compact Lie group, under the map $G \to \mathrm{Conj}(G)$.

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# Example: the Sato-Tate conjecture

Let $E$ be an elliptic curve over a number field $K$. For each prime ideal $\mathfrak{p}$ of $K$ at which $E$ has good reduction, form the normalized characteristic polynomial of Frobenius

$$\overline{L}_{\mathfrak{p}}(E,T) = 1 - \frac{a_{\mathfrak{p}}}{\sqrt{\operatorname{Norm}\mathfrak{p}}}T + T^2.$$

This polynomial defines a unique class $g_{\mathfrak{p}} \in \operatorname{Conj}(\mathrm{SU}(2))$.

**Conjecture (known for $K$ totally real using potential automorphy)**

*If $E$ does not have complex multiplication, then the $\overline{L}_{\mathfrak{p}}(E,T)$ are equidistributed for the image of Haar measure on* $\mathrm{SU}(2)$.

If $E$ has CM, one gets a similar theorem with $\mathrm{SU}(2)$ replaced by $\mathrm{SO}(2)$ if the CM is defined over $K$ or $N(\mathrm{SO}(2))$ if not.

# Abelian varieties

Let $A$ be an abelian variety of dimension $g > 0$ over a number field $K \subset \mathbb{C}$. The *algebraic Sato-Tate group* of $A$ is the $\mathbb{Q}$-algebraic subgroup $\mathrm{AST}(A)$ of

$$\mathrm{Aut}(H^1(A_{\mathbb{C}}^{\mathrm{top}}, \mathbb{Q}), \cup) \cong \mathrm{Sp}(2g, \mathbb{Q})$$

consisting of elements which act on absolute Hodge cycles as some element of $G_K$. The *Sato-Tate group* is a maximal compact subgroup $\mathrm{ST}(A)$ of $\mathrm{AST}(A)_{\mathbb{C}}$.

The connected part of $\mathrm{ST}(A)$ determines the Mumford-Tate group and vice versa. The component group of $\mathrm{ST}(A)$ receives a continuous homomorphism from $G_K$.

Assume the Mumford-Tate conjecture for $A$. For each prime ideal $\mathfrak{p}$ of $K$ at which $A$ has good reduction, $\mathrm{ST}(A)$ contains a distinguished conjugacy class $g_{\mathfrak{p}}$ with characteristic polynomial equal to the normalized characteristic polynomial of Frobenius $\overline{L}_{\mathfrak{p}}(A, T)$.

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# When endomorphisms rule the world

In many cases, the Mumford-Tate group is determined entirely by endomorphisms of $A_{\mathbb{C}}$ (which define absolute Hodge 2-cycles). For instance, this occurs whenever $g \leq 3$.

In this case, the component group equals the Galois group of the minimal field of definition of the endomorphisms of $A_{\mathbb{C}}$.

Moreover, $\mathrm{ST}(A)$ determines the *real* endomorphism algebra of $A_{\mathbb{C}}$ and its $G_K$-action, and vice versa. Warning: this is not true with *real* replaced by *rational*! For instance, $\mathrm{ST}(A)$ does not detect whether or not $A$ is absolutely simple.

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# A classification for abelian surfaces

**Theorem (Fité–K–Rotger–Sutherland)**

*As A varies over abelian surfaces over number fields,* $\mathrm{ST}(A)$ *runs over exactly* 52 *conjugacy classes of subgroups of* $\mathrm{USp}(4)$. *Of these, exactly* 34 *classes occur over* $\mathbb{Q}$.

The most complicated part of the classification occurs in CM cases, when the connected part of $\mathrm{ST}(A)$ equals $\mathrm{U}(1)$. For example, the Jacobian of

$$y^2 = x^6 - 5x^4 + 10x^3 - 5x^2 + 2x - 1$$

has Sato-Tate group an extension of $\mathrm{U}(1)$ by $\mathrm{C}_2 \times \mathrm{S}_4$.

One expects (and can verify numerically using moment statistics and heavy computations) equidistribution of the $g_{\mathfrak{p}}$ in $\mathrm{ST}(A)$, but this seems hopeless to prove in the generic case $\mathrm{ST}(A) = \mathrm{USp}(4)$.

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# Towards a classification for abelian threefolds

The complete classification of Sato-Tate groups for abelian threefolds is not yet known. Some partial results are known in the case where the connected part of $\mathrm{ST}(A)$ equals $\mathrm{U}(1)$, which is expected to occupy the bulk of the classification.

For instance, twists of the Jacobians of the Fermat and Klein quartics

$$x^4 + y^4 + z^4 = 0, \qquad x^3y + y^3z + z^3x = 0$$

produce groups of order 96 and 168.

However, there is also a subgroup of $\mathrm{PGL}_3(\mathbb{C})$ of order 216 (the *Hessian group*) which has not yet been ruled out as a component group. It cannot occur by twisting a curve because $216 > 168 = 84(g-1)$.

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# Some other motives

One can also define Sato-Tate groups for other motives (again using absolute Hodge cycles). To get a good theory, one must assume certain motivic conjectures analogous to the Mumford-Tate conjecture.

For example, consider motives of weight 3 with Hodge numbers $h^{0,3} = h^{1,2} = h^{2,1} = h^{3,0} = 1$. By group theory, we can limit the Sato-Tate group to 26 possibilities, but only 23 are known to occur. For example, symmetric cubes of elliptic curves give SU(2), SO(2), $N$(SO(2)). Many split cases arise from sums/products of modular forms. The case USp(4) occurs generically in the Dwork pencil

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# Fields of definition of endomorphisms

For $A$ an abelian variety over $K$ of dimension $g$, the minimal field of definition of the endomorphisms of $A_{\mathbb{C}}$ is a finite Galois extension $L$ of $K$. Silverberg gives an upper bounds for $[L : K]$ based on orders of symplectic groups over finite fields. An easy corollary is

$$[L : K] \leq 2(9g)^{2g}.$$

In principle, analysis of Sato-Tate groups can be used to improve these bounds. For example, for $g = 2$, the optimal bound is $[L : K] \leq 48$.

Question: can one use real Lie group classifications to derive bounds for general $g$ which are sharper than Silverberg's bounds?

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