Towards global (φ, Γ)-modules and comparison isomorphisms

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The comparison isomorphism in p-adic Hodge theory

Let X be a smooth proper scheme over $\mathbb{Q}_p$. The *comparison isomorphism* of p-adic Hodge theory asserts that the étale cohomology group

$$H^i_{\mathrm{et}}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$$

and the algebraic de Rham cohomology group

$$H^i_{\mathrm{dR}}(X, \mathbb{Q}_p)$$

functorially determine each other once extra structures are accounted for (the Galois action on étale cohomology, the Hodge filtration and crystalline Frobenius/monodromy on de Rham cohomology).

The form of the comparison isomorphism was originally described conjecturally by Fontaine (based on ideas of Tate). Constructions have been given by various people (Faltings, Tsuji, Nizioł, Beilinson, Bhatt, Scholze).

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Galois representations and (φ, Γ)-modules

Another key result of p-adic Hodge theory is the equivalence of categories between continuous representations of G_K on finite free $\mathbb{Z}_p$-modules and (φ, Γ)-*modules*. These are finite free modules over a certain ring S equipped with semilinear actions of a single endomorphism φ and a group $\Gamma \cong \mathbb{Z}_p^\times$.

There are multiple options for the ring S giving rise to equivalent categories of (φ, Γ)-modules. One option is $W(F)$ for F the completed perfect closure of $\mathbb{F}_p((T))$ with the actions of φ, Γ being given by

$$\varphi(T) = (1+T)^p - 1, \qquad \gamma(T) = (1+T)^\gamma - 1.$$

(φ, Γ)-modules and the comparison isomorphism

Let X be a smooth proper scheme over $\mathbb{Q}_p$. One can reinterpret the comparison isomorphism by saying that the (φ, Γ)-module corresponding to $H^i_{\mathrm{et}}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$ and the algebraic de Rham cohomology group $H^i_{\mathrm{dR}}(X, \mathbb{Q}_p)$ functorially determine each other once extra structures are accounted for.

The purpose of this talk is to encourage a different interpretation: think of the functor taking X to the (φ, Γ)-module itself as a cohomology theory, from which both de Rham and étale cohomology can be functorially recovered.

This opens the possibility of enhancing the construction of the (φ, Γ)-module in such a way so that it recovers additional structures which do not fit into the framework of the usual comparison isomorphism. We will end with an example of this.

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An alternate base ring for (φ, Γ)-modules

Again, let F be the completed perfect closure of $\mathbb{F}_p((T))$. Let $W^1(F)$ be the subring of $W(F)$ consisting of those $x = \sum_{n=0}^{\infty} p^n [\overline{x}_n]$ for which $p^{-n} |\overline{x}_n| \to 0$ as $n \to \infty$. The ring $W^1(F)$ is stable under φ^{-1} (but not φ) and Γ.

Theorem

The category of (φ, Γ)-modules over $W(F)$ is equivalent to the category of (φ^{-1}, Γ)-modules over $W^1(F)$. Consequently, the latter is equivalent to the category of continuous representations of $G_{\mathbb{Q}_p}$ on finite free $\mathbb{Z}_p$-modules.

(φ^{-1}, Γ)-modules and de Rham cohomology

The ring $W^1(F)$ admits a continuous homomorphism θ to the completion K of $\mathbb{Q}_p(\mu_{p^\infty})$ taking $[1+T]^{p^{-n}}$ to ζ_{p^n}. Let $\mathbf{B}_{\mathrm{dR}}^+$ be the $\ker(\theta)$-adic completion of $W^1(F)$; it is a complete DVR with residue field K. Let $\mathbf{B}_{\mathrm{dR}}$ be the fraction field of $\mathbf{B}_{\mathrm{dR}}^+$.

Theorem (Fontaine plus comparison isomorphism)

Let X be a smooth proper scheme over $\mathbb{Q}_p$. Let M be the (φ^{-1}, Γ)-module over $W^1(F)$ corresponding to $H^i_{\mathrm{et}}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Z}_p)$. Then there is a functorial isomorphism

$$(M \otimes_{W^1(F)} \mathbf{B}_{\mathrm{dR}})^\Gamma \cong H^i_{\mathrm{dR}}(X, \mathbb{Q}_p)$$

under which the $\ker(\theta)$*-adic filtration on the left corresponds to the Hodge filtration on the right.*

Yet another base ring for (φ, Γ)-modules

The maximal ideal of $\mathbf{B}_{\mathrm{dR}}^+$ is generated by

$$t = \log([1+T]) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}([1+T]-1)^n.$$

Let B^+ be the copy of $\mathbb{Q}(\mu_{p^\infty})[\![t]\!]$ inside $\mathbf{B}_{\mathrm{dR}}^+ \cong K[\![t]\!]$. Let B be the fraction field of B^+.

Let S be the subring of $W^1(F)$ consisting of those x for which for each nonnegative integer n, the image of $\varphi^{-n}(x)$ in $\mathbf{B}_{\mathrm{dR}}^+$ belongs to B^+.

Theorem

The category of (φ^{-1}, Γ)-modules over S is equivalent to the category of (φ^{-1}, Γ)-modules M over $W^1(F)$ equipped with a descent of $M \otimes_{W^1(F)} \mathbf{B}_{\mathrm{dR}}^+$ to B^+.

Rational structures and the comparison isomorphism

Let X be a smooth proper scheme over $\mathbb{Q}$ rather than $\mathbb{Q}_p$. Let M be the (φ^{-1}, Γ)-module over $W^1(F)$ corresponding to $H^i_{\mathrm{et}}(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p)$. We have a natural isomorphism

$$(M \otimes_{W^1(F)} \mathbf{B}_{\mathrm{dR}})^{\Gamma} \cong H^i_{\mathrm{dR}}(X_{\mathbb{Q}_p}, \mathbb{Q}_p).$$

Now the right side carries the extra structure of an isomorphism

$$H^1_{\mathrm{dR}}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong H^i_{\mathrm{dR}}(X_{\mathbb{Q}_p}, \mathbb{Q}_p)$$

of filtered vector spaces. We can thus construct a (φ^{-1}, Γ)-module M_0 over S for which we have a functorial isomorphism

$$(M_0 \otimes_S B)^{\Gamma} \cong H^i_{\mathrm{dR}}(X, \mathbb{Q})$$

of filtered vector spaces, and we can also recover étale cohomology.

Relationship with the previous talk

As in the talk of Chris Davis, one can form the inverse limit of the finite p-typical Witt vector rings over $\mathbb{Q}(\mu_{p^\infty})$ under the Frobenius maps, then pick out the overconvergent subring. One gets the subring of $W^1(F)$ consisting of those x for which for each nonnegative integer n, $\theta(\varphi^{-n}(x)) \in \mathbb{Q}(\mu_{p^\infty})$.

This is bigger than our desired ring S because we need to impose conditions modulo powers of t, not just modulo t.

Question: is there a more Witt-theoretic way to describe S? The answer to this question may involve the differentials in the de Rham-Witt complex, not just Witt vectors (which form the degree 0 part of de Rham-Witt).

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Next steps

In joint work with Chris Davis, we are trying to extend this construction to capture other cohomological invariants within the theory of (φ, Γ)-modules. Especially, given a smooth proper scheme over $\mathbb{Q}$, one would like to combine in a natural way the various (φ, Γ)-modules coming from base extensions to $\mathbb{Q}_p$ for various primes p.

We expect this to involve the de Rham-Witt complex in a natural way. A preliminary step would be to take a smooth proper scheme over $\mathbb{Z}_p$ and relate its (φ, Γ)-modules (or the closely related *Wach modules* or *Breuil-Kisin modules*) to the *absolute de Rham-Witt complex* of Hesselholt-Madsen.

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