Hypergeometric L-functions in average polynomial time

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The MIT campus sits on the traditional unceded territory of the Wampanoag Nation; we acknowledge the painful history of genocide and forced removal from this territory. The UCSD campus sits on the ancestral homelands of the Kumeyaay Nation; the Kumeyaay people continue to have an important and thriving presence in the region.

Arithmetic L-functions: examples

Given a smooth proper scheme X over a number field K, one can define **(incomplete) arithmetic L-functions**. These are Dirichlet series defined by products indexed by finite places of K at which X has good reduction.

Example

Take $X = \operatorname{Spec} K$. Then one gets the Dedekind zeta function

$$\zeta_K(s) = \prod_{\mathfrak{p}} (1 - \operatorname{Norm}(\mathfrak{p})^{-s})^{-1}.$$

Example

Let X be an elliptic curve over K. Then one of the L-functions is

$$L(X, s) = \prod_{\mathfrak{p}} (1 - a_{\mathfrak{p}} \operatorname{Norm}(\mathfrak{p})^{-s} + \operatorname{Norm}(\mathfrak{p})^{1-2s})^{-1}$$

where $a_{\mathfrak{p}}$ is the trace of Frobenius of $X_{\mathfrak{p}}$ (the mod-$\mathfrak{p}$ reduction of X).

Arithmetic L-functions: a general definition

In general, for $i \in \{0, \dots, 2\dim(X)\}$, one gets an L-function whose factor at $\mathfrak{p}$ is $L_i(\mathrm{Norm}(\mathfrak{p})^{-s})^{-1}$, where L_i appears in the zeta function of $X_\mathfrak{p}$:

$$Z(X_\mathfrak{p}, T) = \frac{L_1(T) \cdots L_{2\dim(X)-1}(T)}{L_0(T) \cdots L_{2\dim(X)}(T)}.$$

On the previous slide, for $X = \mathrm{Spec}\, K$ we took $i = 0$; for X an elliptic curve we took $i = 1$.

These are expected to have analytic continuation/functional equation after completing the product so that it has one factor for each **finite or infinite** place of K. (Factors at infinite places involve the Gamma function.)

There is a rich theory of **special values** of arithmetic L-functions, including the Dirichlet class number formula, the Birch–Swinnerton-Dyer conjecture, and conjectures of Bloch–Kato, Deligne, Beilinson, etc.

Arithmetic L-functions in the LMFDB

In general, a single L-function can arise in various ways. E.g., isogenous elliptic curves, or abelian varieties, have the same L-function (and conversely by Tate–Faltings).

There are other ways to construct arithmetic L-functions for which there is not a distinguished "geometric origin". For example, any weight-2 rational eigenform for $\Gamma_0(N)$ has an L-function matching some elliptic curve over $\mathbb{Q}$ (Eichler–Shimura), but the latter is only determined up to isogeny.

A primary goal of the L-Functions and Modular Forms Database is to tabulate arithmetic L-functions with diverse discrete parameters (degree, weight, Hodge numbers). This paper is part of a project to add **hypergeometric L-functions**, which provide examples with assorted parameters; see the LMFDB beta site.

Hypergeometric data

A **hypergeometric datum** of degree r consists of two disjoint tuples $(\alpha_1, \dots, \alpha_r), (\beta_1, \dots, \beta_r)$ over $\mathbb{Q} \cap [0,1)$ which are each **balanced**: the multiplicity of any reduced fraction depends only on its denominator. Later we will consider the example

$$\alpha = (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{4}),\ \beta = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3}).$$

To each such datum, we can define a family of arithmetic L-functions of degree r over $\mathbb{Q}$ parametrized by $z \in \mathbb{Q} \setminus \{0,1\}$. The primes p of bad reduction have the following forms.

- p is **wild** if $\gamma \notin \mathbb{Z}_p$ for some $\gamma \in \alpha \cup \beta$ (e.g., 2 and 3 in our example).
- p is **tame** if it is not wild, and either $z \notin \mathbb{Z}_p^\times$ or $z - 1 \notin \mathbb{Z}_p^\times$.

This L-function is associated to a specific scheme defined in terms of $(\alpha, \beta), z$. However, there is no **distinguished** choice of this scheme.

Trace formulas

In order to add an L-function to the LMFDB, we need the first X coefficients of the Dirichlet series, for X on the order* of 2^{24}. It is sufficient to get the prime-power coefficients, as the others can be recovered using unique factorization.

The Euler factor at a prime p can be interpreted as the reverse charpoly of a matrix F_p. To get the desired Dirichlet coefficients, it suffices to compute the trace of F_p^f for all prime powers $q = p^f \leq X$. Note that for any fixed f, we need $q = p^f$ for $p \leq X^{1/f}$.

In similar situations, this is done by constructing F_p from a Weil cohomology theory (étale or p-adic). In this case, we instead use a direct trace formula based on **finite hypergeometric sums** (Greene, Katz, McCarthy, Beukers–Cohen–Mellit), plus the **Gross–Koblitz formula** for Gauss sums in terms of p-adic functions (Cohen–Rodriguez Villegas).

*The precise cutoff depends on the **conductor** of the L-function.

A preview of the formula

For $q = p^f$, the trace of F_p^f is given by

$$H_q\left(\begin{matrix}\alpha \\ \beta\end{matrix}\middle| z\right) := \frac{1}{1-q}\sum_{m=0}^{q-2}(-p)^{\eta_m(\alpha)-\eta_m(\beta)}q^{D+\xi_m(\beta)}\left(\prod_{j=1}^{r}\frac{(\alpha_j)_m^*}{(\beta_j)_m^*}\right)[z]^m,$$

where $(\gamma)_m^*$ is a p-adic variant of the Pochhammer symbol $(\gamma)_m = \gamma(\gamma+1)\cdots(\gamma+m-1)$ defined using the Morita p-adic Gamma function (see §2.1); $[z]$ is the multiplicative lift† of the reduction of z modulo p; and η_m, ξ_m are discrete invariants independent of z (see §2.2).

More discussion of this formula will take place in the live session. For the moment, note that the sum has $q-1$ terms.

†Commonly called the **Teichmüller lift**, but I recommend phasing out this eponym.

Amortization over primes

The trace formula is implemented in `Magma` and `Sage`. For each q its complexity is $O(q)$ (with small constants), so computing all Dirichlet coefficients up to X incurs complexity $O(X^2)$ (modulo log factors), dominated by the case $f = 1$. (The remaining cases add up to $O(X^{3/2})$.)

However, the shape of the formula makes it feasible to amortize this complexity over q, so that the complexity for each trace is $\mathrm{polylog}(X)$. We establish a partial result, restricting to $f = 1$ and reducing modulo p.

Theorem (Theorem 5.26 of the paper; details in §4, §5.1, §5.2)

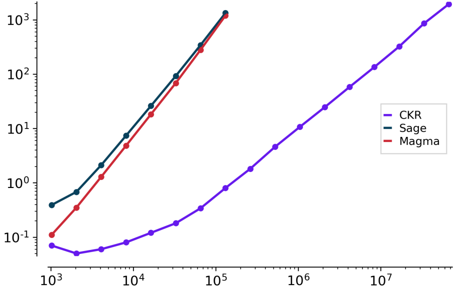
We exhibit an algorithm to compute $H_p\left(\begin{matrix}\alpha\\\beta\end{matrix}\middle| z\right) \pmod p$ *for all primes* $p \leq X$. *For fixed* α, β, z, *the complexity is* $O(X)$ *modulo log factors.*

We have implemented this in Sage/Cython (plus C code by Sutherland for remainder forests). The change from $O(X^2)$ to $O(X)$ appears clearly...

Timings

In this example $\alpha = (\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}), \beta = (\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}), z = \frac{1}{5}$. This L-function has weight 1, so $H_p\left(\begin{matrix}\alpha \\ \beta\end{matrix}\middle| z\right)$ is uniquely determined by its reduction mod p. (See §5.4 of the paper for more implementation details, and §5.5 for a worked example.)

X	Amortized	Sage	Magma
2^{10}	0.07s	0.39s	0.11s
2^{11}	0.05s	0.68s	0.35s
2^{12}	0.06s	2.12s	1.29s
2^{13}	0.08s	7.39s	4.83s
2^{14}	0.12s	26.0s	18.2s
2^{15}	0.18s	92.3s	68.4s
2^{16}	0.34s	343s	280s
2^{17}	0.80s	1328s	1190s



X	2^{18}	2^{19}	2^{20}	2^{21}	2^{22}	2^{23}	2^{24}	2^{25}	2^{26}
Amortized	1.81s	4.59s	10.7s	24.6s	58.0s	135s	322s	857s	1948s

Remainder trees

The key to amortizing is to reduce to subproblems of the following form: given a square matrix M over $\mathbb{Z}[x]$ and a function $k(p)$, compute

$$M(0) \cdots M(k(p) - 1) \pmod{p}$$

for all primes p in some arithmetic progression.

This can be done using **remainder trees/forests**, inspired by the fast Fourier transform. For more details, see Sutherland's talk (and §3).

We take M to be 2×2 triangular; the diagonal entries capture factorial-like products and the off-diagonal captures summation (see §4, §5.1, §5.2).

The mod-p restriction can probably be removed; this would simplify computing Dirichlet coefficients up to X from $O(X^2)$ to $O(X^{3/2})$. The restriction to prime Frobenius traces is subtler (see §2.2.2, §6.1, §6.2).

More details about these points will be given in the live session.

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