Crystals, Crew's conjecture, and cohomology

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These are notes from three lectures given by the author at the University of Arizona on May 8 and 9, 2003, describing some recent progress in p-adic (rigid) cohomology of algebraic varieties. Lectures 1 and 2 are completely independent, while Lecture 3 depends on both of the others.

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Notation

Throughout, we use the following notation.

- k be a field of characteristic $p > 0$.
- K is a field of characteristic 0, complete with respect to a discrete valuation, with residue field k.
- $\mathcal{O}$ is the ring of integers of K.
- $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}$.
- $v(x)$ is the valuation of $x \in K$, normalized so that $v_p(p) = 1$.
- $\sigma : K \to K$ is a continuous automorphism inducing the p-power (absolute) Frobenius on k.

1 Crystals

Crystals, or more properly isocrystals, are the p-adic analogues of locally constant sheaves in ordinary topology, locally free sheaves in sheaf cohomology, lisse sheaves in étale cohomology, or local systems in de Rham cohomology. The closest analogy is the last one: when considering algebraic de Rham cohomology of a smooth affine variety over a field of characteristic 0, local systems are simply finite locally free modules over the coordinate ring, equipped with an integrable connection.

1.1 Convergent isocrystals on smooth affines

Let $X = \operatorname{Spec} \overline{A}$ be a smooth affine scheme of finite type over k. By a theorem of Elkik, there exists a smooth affine scheme $\tilde{X}$ of finite type over $\mathcal{O}$ with $\tilde{X} \times_{\mathcal{O}} k = X$. We will work not with the coordinate ring of $\tilde{X}$, which depends on the choice of $\tilde{X}$, but with its p-adic completion $\widehat{A}$, which by a theorem of Grothendieck is unique up to *noncanonical* isomorphism; we call $\widehat{A}$ a *complete lift* of $\overline{A}$. (The noncanonicality of complete lifts suggests the use of the indefinite article here.) We can write $\widehat{A} = \mathcal{O}\langle x_1, \dots, x_n\rangle/(f_1, \dots, f_m)$ for some n and f_i, where $\mathcal{O}\langle x_1, \dots, x_n\rangle$ is the set of power series convergent for $|x_1|, \dots, |x_n| \leq 1$. (The latter is the p-adic completion of $\mathcal{O}[x_1, \dots, x_n]$.)

Let I be the ideal of the completed tensor product $\widehat{A}[\frac{1}{p}] \hat{\otimes}_K \widehat{A}[\frac{1}{p}]$ which is the kernel of the multiplication map $a \otimes b \mapsto ab$. We then define $\Omega^1 = I/I^2$, which is clearly an $\widehat{A}[\frac{1}{p}]$-module. If $\widehat{A} \cong \mathcal{O}\langle x_1, \dots, x_n\rangle$, this is the quotient of the free $\widehat{A}[\frac{1}{p}]$-module generated by $dx_1, \dots, dx_n$ by the submodule generated by $df_1, \dots, df_m$. Let Ω^i be the i-th exterior power of Ω^1 over $\widehat{A}[\frac{1}{p}]$.

A *convergent isocrystal* over X is a finite locally free $\widehat{A}[\frac{1}{p}]$-module M equipped with an integrable connection $\nabla : M \to M \otimes_{\widehat{A}[\frac{1}{p}]} \Omega^1$. The integrable connection condition means that ∇ is an additive, K-linear, homomorphism satisfying the Leibniz rule

$$\nabla(am) = a\nabla(m) + m \otimes da \qquad (a \in \widehat{A}[\frac{1}{p}], m \in M)$$

such that the maps

$$0 \to M \to M \otimes \Omega^1 \to M \otimes \Omega^2 \to \cdots$$

induced by ∇ form a complex of K-vector spaces. The condition that ∇ is convergent is a bit technical, but here's the ideal: if $t_1, \dots, t_n$ are local coordinates on X, then contracting ∇ with $\frac{\partial}{\partial t_j}$ gives a map $D_j : M \to M$. (Don't forget that $\frac{\partial}{\partial t_j}$ depends on the entire choice of coordinates, not just on t_j!) The maps D_j all commute with each other because ∇ is integrable. The convergence condition states that for $m \in M$, $a_1, \dots, a_n \in \widehat{A}$ with $|a_j| < 1$, and $c_I \in \widehat{A}$ for each n-tuple $I = (i_1, \dots, i_n)$ of nonnegative integers, the series

$$\sum_I c_I a_1^{i_1} \cdots a_n^{i_n} \frac{D_1^{i_1} \cdots D_n^{i_n}(M)}{i_1! \cdots i_n!}$$

converges to an element of M.

The complex

$$0 \to M \to M \otimes \Omega^1 \to M \otimes \Omega^2 \to \cdots$$

we wrote down earlier, in which all maps are induced by ∇, is the *de Rham complex* of M, and its cohomology is the *convergent cohomology of X with coefficients in M*.

That last definition should give some pause, as we have already note that the ring $\widehat{A}$ is only determined by X up to noncanonical isomorphism. However, this is not a problem: given an isomorphism $\iota : \widehat{A} \to \widehat{A}$ which reduces to the identity modulo $\mathfrak{m}$, the maps $\mathrm{id}_{\widehat{A}}$ and ι on the de Rham complex of the trivial isocrystal are homotopic. This yields a canonical isomorphism $\iota^* M \to M$ for any convergent isocrystal M. More generally, if $X \to Y$ is a morphism of smooth affine schemes, $\overline{A}$ and $\overline{B}$ are the coordinate rings of X and Y, respectively, and $\widehat{A}$ and $\widehat{B}$ are complete lifts, then there exists a K-algebra homomorphism $f : \widehat{B} \to \widehat{A}$ which induces the correct homomorphism from $\overline{B}$ to $\overline{A}$, and the pullback f^*M is independent of the choice of f up to canonical isomorphism. (The homotopy on de Rham complexes arises from the fact that any two lifts of the map are p-adically "close together", and so one can be continuously deformed into the other.) Minhyong Kim suggests a better way to formulate this (by analogy with crystalline cohomology): form the category of triples $(X, \widehat{A}, M)$, where X is a smooth affine k-scheme of finite type, $\widehat{A}$ is a complete lift of X, M is a convergent isocrystal over $\widehat{A}$, and morphisms are exactly morphisms on the underlying schemes. Then this category is fibred over the category of smooth affine k-schemes of finite type; that means precisely that there are pullback functors along morphisms.

Convergent cohomology turns out not to be very useful: for instance, the convergent cohomology of the trivial isocrystal on $\mathbb{A}^1$ is not finite dimensional, because the differential

$d : K\langle t\rangle \to \Omega^1$ is far from being surjective. For instance, for any $a_i \in k$ of which infinitely many are nonzero, the differential $\sum_i a_i p^i t^{p^i-1}\,dt$ is not exact.

One note on terminology: the term "isocrystal" is short for "crystal up to isogeny". It arises from the fact that the category of isocrystals on X is equivalent to a full subcategory of the isogeny category of crystals of $\mathcal{O}_{\mathrm{crys}} \otimes K$-modules on X [O]. (The isomorphism category is given by working with modules over $\widehat{A}$ rather than $\widehat{A}[\frac{1}{p}]$.) One might also expect to have a similar equivalence between finite locally free $\widehat{A}$-modules with convergent integrable connection and a full subcategory of the isomorphism category of torsion-free crystals of $\mathcal{O}_{\mathrm{crys}}$-modules; this is easy to verify locally, but I'm not sure if it holds globally. In any case, one can fruitfully exploit the connection to crystalline cohomology (e.g., in Berthelot's proof of finite dimensionality of rigid cohomology with constant coefficients [Be]), but we will not do so here, instead remaining entirely within the rigid setting.

1.2 Overconvergent isocrystals on smooth affines

While convergent isocrystals arise naturally in geometry, they are not well suited for cohomology; as already noted, even in simple examples the resulting cohomology is infinite-dimensional. We thus introduce a more refined notion, that of an overconvergent isocrystal, which gives a better cohomology theory.

An algebra R equipped with a nonarchimedean absolute value $|\cdot|$ is *weakly complete* if for any $f_1, \dots, f_n \in R$ with $|f_i| < 1$, and any $c_I \in R$ for each n-tuple $I = (i_1, \dots, i_n)$ of nonnegative integers with $|c_I| \geq 1$, the sum

$$\sum_I c_I f_1^{i_1} \cdots f_n^{i_n}$$

converges under $|\cdot|$ to an element of R. A complete algebra is weakly complete, but not vice versa. For instance, the algebra $\mathcal{O}\langle x_1, \dots, x_n\rangle^\dagger$ of series for which there exists an $\eta > 1$ so that the series converges for $|x_1|, \dots, |x_n| \leq \eta$ is weakly complete but not complete.

Let $X = \operatorname{Spec} \overline{A}$ be a smooth affine scheme of finite type over k. By the theorem of Elkik mentioned before, there is a smooth finitely generated $\mathcal{O}$-algebra A with $A \otimes_{\mathcal{O}} k \cong \overline{A}$. It turns out that the weak completion $A^\dagger$ of A is again unique up to noncanonical isomorphism. We call $A^\dagger$ a *weakly complete lift* of $\overline{A}$. We can write $A^\dagger = \mathcal{O}\langle x_1, \dots, x_n\rangle^\dagger/(f_1, \dots, f_m)$ for some n and f_i; in particular, $A^\dagger$ is noetherian.

One defines Ω^1 as in the complete case: let I be the ideal of the weakly completed tensor product $A^\dagger[\frac{1}{p}] \otimes_K^\dagger A^\dagger[\frac{1}{p}]$ which is the kernel of the multiplication map $a \otimes b \mapsto ab$, and put $\Omega^1 = I/I^2$. Likewise, one defines an *overconvergent isocrystal* by simply replacing complete lifts with weakly complete lifts everywhere in the definition. Again, the resulting category is independent of the choice of the lift. The de Rham cohomology of an isocrystal M in this case is called the *overconvergent cohomology*, or more commonly the *rigid cohomology, of X with coefficients in M*. For example, the rigid cohomology of $\mathbb{A}^1$ with coefficients in the trivial isocrystal is finite dimensional and has sensible Betti numbers: H^0 is one-dimensional and all other spaces are zero-dimensional.

It is clear that there is a faithful functor from overconvergent to convergent isocrystals, by tensoring up from a weakly complete lift to its completion. More on this functor shortly.

1.3 Frobenius structures

It seems difficult to prove anything about isocrystals in the abstract; the disconnectedness of the p-adic topology makes analytic continuation much more delicate. A fundamental insight of Dwork is that isocrystals that arise from geometric situations come with an extra structure, the so-called Frobenius structure, that rigidifies the isocrystal in the absence of analytic continuation.

Let $X = \operatorname{Spec} \overline{A}$ be a smooth affine scheme of finite type over k and let A be either a complete or weakly complete lift of $\overline{A}$. Let $\sigma : A \to A$ be a continuous homomorphism lifting the p-power map modulo $\mathfrak{m}$ and restricting to the chosen map $\sigma : K \to K$. Then for any (convergent or overconvergent) isocrystal M over X, $\sigma^* M$ is again an isocrystal; a *Frobenius structure* is an isomorphism $F : \sigma^* M \to M$ of isocrystals. By the usual homotopy construction, $\sigma^* M$ is independent of the choice of σ up to canonical isomorphism. A (convergent or overconvergent) isocrystal equipped with a (convergent or overconvergent) Frobenius structure is called a *(convergent or overconvergent) F-isocrystal*.

The presence of Frobenius structure facilitates numerous computations involving isocrystals. For example, any convergent morphism between overconvergent F-isocrystals is overconvergent, i.e., the functor from overconvergent to convergent F-isocrystals is fully faithful [K2]. (In particular, if a convergent F-isocrystal can be made overconvergent, it can be made so in only one way.) Also, a convergent F-isocrystal which becomes overconvergent upon restriction to a dense open subset is overconvergent [K6]. Both results are expected to on isocrystals without Frobenius structure, but in that case they are not known. (Also, both results hold on smooth but not necessarily affine schemes, and are believed to hold even on nonsmooth schemes; Tsuzuki's rigid cohomological descent probably allows a reduction to the smooth case, but this has not yet been checked.)

Unfortunately, the benefits of having a Frobenius structure are not available when one is trying to establish that an isocrystal has such a structure. For example, a conjecture of Tsuzuki would imply that a convergent Frobenius structure on an overconvergent isocrystal is itself overconvergent, but this has not been independently established.

Incidentally, to verify that a given module is an F-isocrystal, it suffices to verify the fact that $\nabla \circ \nabla = 0$ to get integrability; the convergence condition is automatic by "Dwork's trick". This is handy in examples, since the convergence condition is a bit annoying to verify directly. For instance, let $f : E \to X$ be a family of elliptic curves over a smooth base. Then there is a rank 2 overconvergent F-isocrystal $R^1 f_* \mathcal{O}$ on X whose fibre at $x \in X$ is the cohomology of the elliptic curve E_x: producing the overconvergent module and the Frobenius is not so hard, and the convergence condition is then free. Incidentally, if each fibre is an ordinary elliptic curve, then $R^1 f_* \mathcal{O}$ admits a rank 1 subobject in the convergent category (the "unit-root subcrystal"), but this object is *not* overconvergent.

1.4 Isocrystals on nonsmooth/nonaffine schemes

For smooth but not necessarily affine varieties, convergent and overconvergent isocrystals can be constructed by glueing: they can be described by giving isocrystals on an affine cover and glueing isomorphisms on the overlaps which satisfy the cocycle condition on triple intersections. The cohomology of such an isocrystal can be computed (or even defined!) using the hypercohomology spectral sequence.

For affine but not necessarily smooth varietes, a different description is needed. Several are possible, but my favorite is the following one given by Grosse-Klönne [G1]. (Actually, his description is more general, but it implies that this one is correct.) Let $X = \operatorname{Spec} \overline{A}$ be an affine (but not necessarily smooth) scheme of finite type over k, and choose a presentation $\overline{A} \cong k[\overline{x}_1, \dots, \overline{x}_n]/(\overline{f}_1, \dots, \overline{f}_m)$. Choose lifts f_i of $\overline{f}_i$ into $\mathcal{O}[x_1, \dots, x_n]$, form the ring $\mathcal{T}_{m,n}$ of power series in $x_1, \dots, x_n, y_1, \dots, y_m$ which converge for $|x_i| \leq 1$ and $|y_i| < 1$, and put

$$A_{\mathrm{con}} = \mathcal{T}_{m,n}/(y_1 - f_1, \dots, y_m - f_m).$$

Then a convergent isocrystal on X is a finitely presented, locally free A_{con}-module equipped with an integrable connection (for an appropriate definition of Ω_1). Beware that now the ring A_{con} is not independent of X even up to noncanonical isomorphism; but it is independent up to "homotopy equivalence", so one gets a canonical category of convergent isocrystals.

To get overconvergent isocrystals, one passes to the subring $\mathcal{T}^{\dagger}_{m,n}$ of series such that for any $\delta < 1$, there exists $\eta > 1$ (depending on the series and on δ) such that the series converges for $|x_i| \leq \eta$ and $|y_j| \leq \delta$. (Once a series belongs to $\mathcal{T}^{\dagger}_{m,n}$, it is enough to check this additional condition for a single δ.) Then one works with modules over

$$A_{\mathrm{ocon}} = \mathcal{T}^{\dagger}_{m,n}/(y_1 - f_1, \dots, y_m - f_m)$$

instead of A_{con}. In both cases, the module Ω_1 ends up being freely generated by $dx_1, \dots, dx_n$, and one defines de Rham cohomology as before. (And again, one can glue to define convergent or rigid cohomology on nonsmooth nonaffines.)

2 Crew's conjecture

Although a full theory of p-adic vanishing cycles has not yet been developed, it has become clear how to interpret the notion of "the p-adic local monodromy of an isocrystal on a curve", thanks to the work of Crew. In his work on this subject (e.g., see [Cr]) arose a conjecture about p-adic differential equations; in particular, under this conjecture, Crew proved finite dimensionality of the rigid cohomology of an overconvergent F-isocrystal on a curve. The conjecture was reformulated in entirely local guise (as we present it here) by Tsuzuki [T2], but the name "Crew's conjecture" is the most common. (Now that it has been resolved, it is also known as the "p-adic local monodromy theorem".)

2.1 The Robba ring

Let $\mathcal{R}$ be the set of formal sums $\sum_{n=-\infty}^{\infty} c_n t^n$, with $c_n \in K$, such that

$$\liminf_{n \to -\infty} \frac{v(c_n)}{-n} > 0, \qquad \liminf_{n \to \infty} \frac{v(c_n)}{n} \geq 0.$$

Then $\mathcal{R}$ forms a ring under series convolution, called the *Robba ring*. Its elements may be interpreted as Laurent series which converge on some open annulus of outer radius 1.

A theorem of Lazard implies that the Robba ring is a Bézout ring: every finitely generated ideal is principal. In particular, every finitely presented projective module is free.

Let $\mathcal{R}^{\text{int}}$ be the subring of $\mathcal{R}$ whose coefficients belong to $\mathcal{O}$; then $\mathcal{R}^{\text{int}}$ is a henselian (but noncomplete) discrete valuation ring with residue field $k((t))$. Recall that $\sigma : K \to K$ lifts the p-power Frobenius on k. Choose an extension $\sigma : \mathcal{R} \to \mathcal{R}$ of σ to $\mathcal{R}$ of the form

$$\sum c_n t^n \mapsto \sum c_n^\sigma (t^\sigma)^n,$$

where $t^\sigma \in \mathcal{R}_K^{\text{int}}$ reduces to t^p.

The Robba ring can be viewed as the limit, over $0 < \rho < 1$, of the subring $\mathcal{R}_\rho$ of series convergent for $\rho < |t| < 1$, and of course any finitely presented module over $\mathcal{R}$ is actually defined over such a subring. One typically must restrict to such a subring to accomplish any analysis (e.g., summing an infinite series) over $\mathcal{R}$. However, one cannot avoid $\mathcal{R}$ entirely, as these subrings are not preserved by any σ: for ρ sufficiently close to 1, σ carries $\mathcal{R}_\rho$ into $\mathcal{R}_{\rho^{1/p}}$.

2.2 (F, ∇)-modules

Let M be a finite free module over $\mathcal{R}$. Crew's conjecture is a structural classification of a certain pair of extra structures on M; these are local analogues of the connection and Frobenius structures on overconvergent F-isocrystals.

A *Frobenius structure* on M is an additive, σ-linear map $F : M \to M$ whose image generates M over $\mathcal{R}_K$. (One would like to say that F is bijective, but that is too strong: that does not even hold for the trivial Frobenius structure given by σ on $\mathcal{R}$ itself.) Here σ-linearity means that $F(rm) = r^\sigma F(m)$ for $r \in \mathcal{R}$. Equivalently, a Frobenius structure on M is an isomorphism of the $\mathcal{R}$-modules $\sigma^* M$ (which looks like M but with the $\mathcal{R}$-action funneled through σ) and M.

A *connection* on M is an additive, K-linear map $\nabla : M \to M \otimes_{\mathcal{R}} \Omega^1_{\mathcal{R}/K}$ satisfying the Leibniz rule $\nabla(rm) = r\nabla(m) + m \otimes dr$ for $r \in \mathcal{R}$ and $m \in M$. Here $\Omega^1_{\mathcal{R}/K}$ is the free $\mathcal{R}$-module generated by dt and $d : \mathcal{R} \to \Omega^1_{\mathcal{R}/K}$ is the derivation

$$\sum c_n t^n \mapsto \left(\sum n c_n t^{n-1} \right) dt.$$

A Frobenius structure F and a connection ∇ on M are said to be *compatible* if F induces an isomorphism of $\sigma^* M$ with M as modules with connection. In other words, the following diagram should commute:

$$\begin{array}{ccc} M & \xrightarrow{\nabla} & M \otimes \Omega^1_{\mathcal{R}/K} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F\otimes d\sigma} \\ M & \xrightarrow{\nabla} & M \otimes \Omega^1_{\mathcal{R}/K} \end{array}$$

where $d\sigma$ is the linearization of σ: $d\sigma(df) = d(f^\sigma)$. An (F,∇)-*module* is a finite free module M over $\mathcal{R}$ equipped with compatible Frobenius and connection structures. (Note that this concept is independent of the choice of σ: the presence of ∇ allows one to "deform" a Frobenius structure with respect to a particular σ into one for another σ.)

2.3 The p-adic local monodromy theorem

An (F,∇)-module is *constant* if it is spanned by the kernel of ∇. An (F,∇)-module is *unipotent* if it admits a filtration by saturated (F,∇)-submodules whose successive quotients are constant.

For L a finite separable extension of $k((t))$, there is a unique finite unramified extension of $\mathcal{R}^{\mathrm{int}}$ with residue field L (because $\mathcal{R}^{\mathrm{int}}$ is henselian); if we call this extension $\mathcal{R}^{\mathrm{int}}_L$, then $\mathcal{R}_L = \mathcal{R} \otimes_{\mathcal{R}^{\mathrm{int}}} \mathcal{R}^{\mathrm{int}}_L$ is also isomorphic to the Robba ring, but with a different series parameter.

We say an (F,∇)-module is *quasi-constant* if it becomes constant over $\mathcal{R}_L$ for some finite separable extension L of $k((t))$, and *quasi-unipotent* if it admits a filtration by saturated (F,∇)-submodules whose successive quotients are constant (or equivalently, if it becose unipotent over $\mathcal{R}_L$ for some finite separable extension L of $k((t))$). Then Crew's conjecture is the following assertion.

Theorem 2.1 (p-adic local monodromy theorem). *Every (F,∇)-module over $\mathcal{R}$ is quasi-unipotent.*

This theorem has been established independently by André [A], Mebkhout [M2], and the speaker [K1]. We will comment on these proofs in the next section.

2.4 The canonical filtrations

The proofs of Crew's conjecture all proceed by first establishing the existence of a canonical filtration for modules over the Robba ring equipped with one of the two structures (Frobenius or connection), then using the other to separate things further. For more details about these proofs, a great reference is Colmez's Seminaire Bourbaki of November 2001 [Co], though unfortunately it is not yet published.

First, suppose M is a finite free $\mathcal{R}$-module equipped with a connection ∇ but no Frobenius. Under a certain technical hypothesis on ∇ (always satisfied in the presence of Frobenius and sometimes otherwise), the p-adic index theorem of Christol and Mebkhout [CM] produces a canonical ascending filtration on M, called the *weight filtration*. (The construction of this filtration involves a formidable p-adic analytic computation, about which we will not comment further.) For M a (F, ∇)-module, the weight filtration admits the following *a posteriori* interpretation (due to Matsuda): for $r \in \mathbb{Q}$ nonnegative, the step M_r of the filtration is the maximal (F, ∇)-submodule which becomes unipotent over some extension of $k((t))$ whose ramification filtration has all its jumps less than or equal to r, in the upper numbering. (This is a bit confusing: we made an extension of the DVR $\mathcal{R}^{\text{int}}$, but that extension is unramified, so has no ramification numbers. It is the residue field extension of $k((t))$ that contributes the ramification numbers.) For instance, M_0 is the maximal (F, ∇)-submodule which becomes unipotent over a tamely ramified extension of $k((t))$. The main difficulty in the construction is first establishing a criterion that determines the jumps in the weight filtration assuming that it exists, then converting that criterion into an actual construction.

The proofs of Crew's conjecture by André and Mebkhout both start from the weight filtration, but proceed differently thereafter. André shows that any filtration that looks group-theoretically like the ramification filtration of a local field (a "Hasse-Arf filtration") must be one, and that the weight filtration in the presence of a Frobenius structure is such a filtration. Mebkhout uses a more computational approach, studying the action of Frobenius explicitly on steps of the weight filtration. (Beware that the weight filtration is not a unipotent filtration: for instance, if M is actually unipotent, the weight filtration has a single jump at weight 0.)

The speaker's proof of Crew's conjecture proceeds differently, by working primarily with Frobenius structures. For M a finite free $\mathcal{R}$-module equipped with a Frobenius structure but no connection, one obtains a canonical ascending filtration on M called the *slope filtration*. (The construction of this filtration also involves a formidable p-adic analytic computation, about which we will not comment further.) For M a (F, ∇)-module and k algebraically closed, the slope filtration admits the following *a posteriori* interpretation: for $r \in \mathbb{Q}$ nonnegative, the step M_r of the filtration is the maximal (F, ∇)-submodule which, over some extension of $\mathcal{R}$, is spanned by elements m such that $F^a m = p^b m$ for some integers a, b with $a > 0$ and $b/a \leq r$. The main difficulty in the construction is first establishing a criterion that determines the jumps in the slope filtration assuming that it exists, then converting that criterion into an actual construction.

With the slope filtration in hand, one knows *a priori* that each successive quotient is an (F, ∇)-module over $\mathcal{R}^{\text{int}}$ whose Dieudonné-Manin slopes are all equal. One can reduce to the case where these slopes are zero (the *unit-root* case); this case of Crew's conjecture has been treated by Tsuzuki [T1]. In fact any such (F, ∇)-module is quasi-constant, not just quasi-unipotent, so the slope filtration itself is a unipotent filtration.

Incidentally, if M is itself defined over $\mathcal{R}^{\text{int}}$ to begin with, the Dieudonné-Manin classification itself produces a filtration of M. (That is the situation in the cohomology application, but not in Berger's construction.) This filtration is not the same as the slope filtration described above! I call the Dieudonné-Manin filtration on M the *generic slope filtration* and the one above the *special slope filtration*, because if you draw the Newton polygons of the two sets of slopes, the special Newton polygon lies on or above the generic Newton polygon and has the same endpoint. In fact, this phenomenon is closely related to the construction of the slope filtration: one passes up to a huge ring (containing the maximal unramified extension of $\mathcal{R}^{\text{int}}$) and establishes the existence of the filtration there by a series of successive approximations, producing modules over various DVRS whose Newton polygons are steadily increasing. When the Newton polygon can be raised no further, one gets the desired filtration but over too large a ring; one must then "descend" the filtration back to $\mathcal{R}$.

2.5 Other applications

Although our interest in Crew's conjecture stems from its relationship with p-adic cohomology, there seem to be other applications. We briefly mention two of them.

Building on work of Charbonnier and Colmez, Berger [Bg] has established a link between (F, ∇)-modules and continuous representations $\mathrm{Gal}(\overline{L}/L) \to \Gamma_n^L(K)$, for L a finite extension of $\mathbb{Q}_p$. One can read off many properties of a representation from its corresponding (F, ∇)-module. For instance, a representation is crystalline (resp. semistable) in the sense of Fontaine if and only if its corresponding (F, ∇)-module is constant (resp. unipotent). In particular, Fontaine's conjecture that every de Rham representation is potentially semistable (conjecture C_{st}) follows from Crew's conjecture. (The de Rham condition is symptomatic of representations that "come from geometry"; those that actually do are forced to be potentially semistable by de Jong's alterations theorem. Presumably all de Rham representations come from geometry, but no assertion even remotely resembling this is known.)

Another application, or more precisely a variation, has been given by Yves André. He is interested in q-difference equations, which may be viewed as deformations of differential equations. (The idea is that the function $g(x) = (f(xq) - f(x))/(q-1)$ tends to $xf'(x)$ as $q \to 1$.) In this language, the natural analogue of Crew's conjecture also holds; since the notion of Frobenius structure is not altered by deformation, the analogue can be deduced using the slope filtration. André and di Vizio believe there is also a weight filtration in this context, but have not yet worked out the details. This analogue leads to a theory of q-rigid cohomology, about which the speaker is presently unable to discourse further.

3 Cohomology

In this lecture, we summarize the proof of the following theorem given in [K3]. At the end we also point out how this theorem can be used to give a p-adic derivation of the Weil conjectures.

Theorem 3.1. *The rigid cohomology of an arbitrary overconvergent F-isocrystal on an arbitrary separated finite type k-scheme is a finite dimensional K-vector space.*

For constant isocrystals, this was proved for smooth schemes by Berthelot [Be] and extended to nonsmooth schemes by Grosse-Klönne [G2]. However, Berthelot's proof is based on de Jong's alterations theorem and a reduction to crystalline cohomology, and does not immediately extend to the general case.

Our proof is closer in spirit to the proof of finite dimensionality for the constant isocrystal on a smooth affine k-scheme given by Mebkhout [M1]; in particular, we perform all of our computations on smooth affines and use excision arguments to reduce everything to the smooth affine case. (The excision arguments handle the case of smooth but not necessarily affine schemes; to get to nonsmooth schemes, one invokes the cohomological descent method of Chiarellotto and Tsuzuki [CT].) More specifically, given a smooth affine variety, we fiber it in curves over a variety of dimension one lower, and use Crew's conjecture (Theorem 2.1) to construct higher direct images of the given isocrystal. Then a Leray spectral sequence allows one to deduce finite dimensionality of cohomology of the original isocrystal from finite dimensionality of cohomology of its direct images.

3.1 More on weakly complete lifts

We want to define the Robba ring over a weakly complete lift, but to do so we must first do a bit more "weakly complete algebra".

Let $A^\dagger$ be a weakly complete lift of the smooth affine finite type k-algebra $\overline{A}$. For any presentation $A^\dagger = \mathcal{O}\langle x_1, \dots, x_n\rangle^\dagger/(f_1, \dots, f_m)$, we can write $A^\dagger$ as the direct limit of the subalgebras $T_n(\rho)/(T_n(\rho) \cap (f_1, \dots, f_m))$ over $\rho > 1$, where $T_n(\rho) \subset \mathcal{O}\langle x_1, \dots, x_n\rangle^\dagger$ is the set of series convergent for $|x_1|, \dots, |x_n| \leq \rho$. We call any such subalgebra a *fringe algebra*; a fringe algebra is complete with respect to its "intrinsic" norm (the supremum norm over $|x_1|, \dots, |x_n| \leq \rho$) but not with respect to the norm on $A^\dagger$.

For $A^\dagger$ a weakly complete lift, the *localization* of $A^\dagger$ at some $f \in A^\dagger \setminus \mathfrak{m}A^\dagger$ is the weak completion of $A^\dagger[f^{-1}]$. If $A^\dagger \cong \mathcal{O}\langle x_1, \dots, x_n\rangle^\dagger/(f_1, \dots, f_m)$, then its localization at f is isomorphic to

$$\mathcal{O}\langle x_1, \dots, x_{n+1}\rangle^\dagger/(f_1, \dots, f_m, f x_{n+1} - 1).$$

3.2 The Robba ring over a weakly complete lift

It is easy to define the Robba ring over a complete lift $\widehat{A}$: simply replace the power series over K by power series over $\widehat{A}[\frac{1}{p}]$ and keep the same convergence condition. If we stopped there, we could only hope to construct higher direct images of an overconvergent F-isocrystal in the category of *convergent* F-isocrystals. To get them in the overconvergent category, we will need to define "the Robba ring over a weakly complete lift".

Let L be the p-adic completion of $\operatorname{Frac} A^\dagger$. We define the Robba ring $\mathcal{R}_{A^\dagger}$ as the subring of the Robba ring over L of series $\sum c_n t^n$ such that for some $r > 0$, the quantities $p^{\lfloor rn \rfloor} c_n$ all belong to some fringe algebra and converge to zero as $n \to \pm\infty$ in the intrinsic norm of that fringe algebra. The condition then holds for any smaller r as well (but the fringe algebra may vary), so the result is indeed a ring.

Again, we define Ω^1 as the free module generated by dt. We may then repeat the definition of (F, ∇)-module over $\mathcal{R}_{A^\dagger}$, as well as the unipotent property. Then one has the following relationship between unipotence over a dagger algebra and over a field (see [K3]).

Proposition 3.2. *Let M be a free (F, ∇)-module over $\mathcal{R}_{A^\dagger}$ which becomes unipotent over the Robba ring of L. Then M is unipotent over $\mathcal{R}_{B^\dagger}$ for some localization $B^\dagger$ of $A^\dagger$.*

3.3 Pushforwards in rigid cohomology

Let $f : X \to Y$ be a morphism of smooth affine finite type k-schemes, and let $B^\dagger \to A^\dagger$ be a corresponding morphism of dagger algebras. Then we define the relative module of differentials $\Omega^1_{A^\dagger/B^\dagger}$ as the quotient of $\Omega^1_{A^\dagger/K}$ by the sub-$A^\dagger$-module generated by db for $b \in B^\dagger$. Given an overconvergent F-isocrystal M on X, we define the higher direct images $R^i f_* M$ as the cohomology of the complex $M \otimes \Omega^{\cdot}_{A^\dagger/B^\dagger}$. These are $B^\dagger$-modules with connection and Frobenius, but may not be finitely generated. One can however prove the following.

Proposition 3.3. *If $f : X \to Y$ is smooth of relative dimension 1, and M is an overconvergent F-isocrystal on X, then after restricting to some open dense subset U of Y, the $R^i f_* M$ become overconvergent F-isocrystals on U.*

(The restriction to U is really necessary: the conclusion implies that the fibrewise cohomologies $H^i(X_y)$ have the same rank for all $y \in U$, which may not hold for all $y \in Y$.)

Sketch of proof: by Theorem 2.1 and Proposition 3.2, one can replace X and Y by finite covers (after shrinking Y) so that X embeds into $\overline{X}$ which is smooth and proper of relative dimension 1 over Y, the complement $Z = \overline{X} \setminus X$ is a union of disjoint sections, and M is unipotent along each component of Z. (More on what this means in a moment.) In this case, one can directly compute the $R^i f_* M$ and see that they are finitely generated over a weakly complete lift of U; this implies the finiteness also back for the original X and Y. (End sketch.)

What it means for M to be unipotent along a component of Z: given a component of Z, one gets an embedding of $A^\dagger$ into $\mathcal{R}^{\mathrm{int}}_{B^\dagger}$ which reduces modulo $\mathfrak{m}$ to the embedding of $\overline{A}$ into its completion along Z. Simple case: if $A = K\langle x \rangle^\dagger$, $B = K$ and Z is the point at infinity in $\mathbb{P}^1$, then one such embedding of A into $\mathcal{R}_K$ is $x \mapsto t^{-1}$, lifting the embedding $K[x] \hookrightarrow K((x^{-1}))$. Of course there are many such embeddings, but whether $M \otimes_{A^\dagger} \mathcal{R}_{B^\dagger}$ is unipotent does not depend on the embedding.

How do these pushforwards help us? They fit into a Leray spectral sequence relating the cohomology of the original isocrystal with the cohomology of the pushforwards. Actually the only case we really need is this one (which we can check "by hand"): if M is an overconvergent F-isocrystal on $X \times \mathbb{A}^1$ and $f : X \times \mathbb{A}^1 \to X$ is the canonical projection, then there are canonical exact sequences

$$H^i(X, R^1 f_* M) \to H^i(X \times \mathbb{A}^1, M) \to H^{i-1}(X, R^1 f_* M)$$

for all i.

3.4 How to put it all together

To sum up, here is a summary of the proof of finiteness of rigid cohomology with coefficients in an overconvergent F-isocrystal.

- We proceed by induction on d, proving that:

(a)$_d$ $H^i(X,M)$ is finite dimensional if X is smooth and $\dim X \leq d$;

(b)$_d$ $H^i_Z(X,M)$ is finite dimensional if X is smooth, Z is a smooth subscheme and $\dim Z \leq d$.

(This is the same strategy adopted by Berthelot in [Be].)

- The fact that (a)$_d$ implies (b)$_d$ follows from the existence of a Gysin isomorphism $H^i_Z(X,M) \to H^{i-2d}(Z,M)(-d)$. Technically, this is only true "generically" (after replacing X by an open dense subscheme) because of liftability hypotheses, but that is good enough (by a bit of excision).

- By the excision exact sequence

$$\cdots \to H^i_Z(X,M) \to H^i(X,M) \to H^i(Z,M) \to \cdots,$$

given (b)$_{d-1}$, to prove (a)$_d$ it suffices to prove it "generically", i.e., after replacing any given X by a suitable open dense subscheme. In particular, we can find such a subscheme which is finite étale over $\mathbb{A}^n$ (yes, *finite* étale! This is a trick peculiar to positive characteristic; see [K5]), and it suffices to work with the pushforward N of M down to $\mathbb{A}^n$ (or an open dense subscheme thereof).

- Now view $\mathbb{A}^n$ as $\mathbb{A}^{n-1} \times \mathbb{A}^1$ with projection f onto $\mathbb{A}^{n-1}$; by Proposition 3.3, after shrinking $\mathbb{A}^{n-1}$ suitably, we get $R^i f_* N$ in the category of overconvergent F-isocrystals on $\mathbb{A}^{n-1}$. By the induction hypothesis, these have finite cohomology, as then does N by the Leray construction. That completes the argument for X smooth.

- For X nonsmooth, we invoke cohomological descent as formulated by Chiarellotto and Tsuzuki [CT]. The existence of the necessary proper hypercovering follows from de Jong's alterations theorem [dJ].

- One can also prove finiteness of cohomology with compact supports by proving Poincaré duality: for X smooth of pure dimension d, one has a canonical perfect pairing

$$H^i(X,M) \times H^{2d-i}_c(X,M^\vee) \to H^{2d}_c(X) \cong K(-d).$$

This is easiest to do for $X = \mathbb{A}^n$; again, excision and induction on dimension do the trick in general. That immediately gives finite dimensionality of cohomology with supports for X smooth; now the excision sequence for cohomology with supports

$$\cdots \to H^i_c(U,M) \to H^i_c(X,M) \to H^i_c(Z,M) \to \cdots$$

yields finite dimensionality in general.

- By similar means, one can obtain the Künneth decomposition: if M_i is an overconvergent F-isocrystal on X_i for $i=1,2$, then $H^j(X_1 \times X_2, M_1 \boxtimes M_2) \cong \sum_{a+b=j} H^a(X_1,M_1) \otimes H^b(X_2,M_2)$, and likewise with supports.

3.5 Rigid "Weil II"

As a postscript, we note that Deligne's "Weil II" theorem, which implies the Weil conjectures, can be reproduced in rigid cohomology. (The analogous assertion in crystalline cohomology had earlier been suggested by Faltings [F], though sans some significant technical details.)

Theorem 3.4. *Let X be a separated $\mathbb{F}_q$-scheme of finite type and M an overconvergent F-isocrystal on X. Suppose that for each closed point $x \in X$ of degree d, F^d acts on the fibre M_x via a linear transformation whose characteristic polynomial has rational (resp. integer) coefficients and complex eigenvalues of absolute value $q^{i/2}$. Then F acts on $H_c^j(X, M)$ via a linear transformation whose characteristic polynomial has rational (resp. integer) coefficients and complex eigenvalues each of absolute value $q^{i+j-\ell}/2$ for some nonnegative integer ℓ (depending on the eigenvalue). If X is smooth and proper, then in fact $\ell = 0$ for all eigenvalues.*

The implication of the Weil conjectures follows because one has a Lefschetz trace formula in rigid cohomology, obtained from a construction of Monsky (based on work of Dwork and Reich). In particular, finite dimensionality of rigid cohomology implies rationality of zeta functions, Poincaré duality implies the functional equation for smooth proper varieties, and the theorem above implies the Riemann hypothesis component.

As in the proof of Theorem 3.1, one can reduce Theorem 3.4 to the case where X is a curve, or in fact where $X = \mathbb{A}^1$. Here instead of imitating Deligne's arguments exactly, we instead follow Laumon's derivation using a geometric Fourier transform for constructible sheaves on $\mathbb{A}^1$. The p-adic analogue of this construction is a Fourier transform on arithmetic $\mathcal{D}$-modules, constructed by Huyghe [H1]. See [K4] for more details.

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