Sato-Tate groups of abelian surfaces

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Throughout this talk, let A be an abelian variety[1] of dimension g over a number[2] field K. Its L-function (in the analytic normalization) is defined for $\mathrm{Re}(s) > 1$ as an Euler product

$$\overline{L}_A(s) = \prod_{\mathfrak{p}} \overline{L}_{A,\mathfrak{p}}(q^{-s})^{-1},$$

where for $\mathfrak{p}$ a prime ideal of norm q at which A has good reduction, the *normalized L-polynomial* $\overline{L}_{A,\mathfrak{p}}(T)$ is a unitary reciprocal monic polynomial over $\mathbb{R}$ of degree $2g$. (I ignore what happens at bad reduction primes.)

This L-function is an example of a *motivic L-function*. **From now on, let us assume that such L-functions have meromorphic continuation and functional equation as expected.** (No need to assume RH unless you want power-saving error terms later.)

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Distribution of normalized L-polynomials

Let $\mathrm{USp}(2g)$ be the *unitary symplectic group*. The characteristic polynomial map defines a bijection between $\mathrm{Conj}(\mathrm{USp}(2g))$ and the set of unitary reciprocal monic real polynomials of degree $2g$.

Theorem (conditional!)

The classes in $\mathrm{Conj}(\mathrm{USp}(2g))$ *corresponding to the* $\overline{L}_{A,\mathfrak{p}}(T)$ *are equidistributed with respect to the image of Haar measure on some compact subgroup* $\mathrm{ST}(A)$ *of* $\mathrm{USp}(2g)$. *(The "generic case" is* $\mathrm{ST}(A) = \mathrm{USp}(2g)$.*)*

Concretely, this means that limiting statistics on normalized L-polynomials (e.g., the distribution of a fixed coefficient) can be computed using the corresponding statistics on random matrices in $\mathrm{ST}(A)$. For examples, see

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Distribution of normalized L-polynomials (contd.)

The previous theorem can be made more precise in two ways.

- One can specify the group $\mathrm{ST}(A)$ explicitly in terms of the arithmetic of A. We call it the *Sato-Tate group* of A.
- Using the right definition of $\mathrm{ST}(A)$, one (conjecturally) gets specific classes in $\mathrm{Conj}(G)$, rather than $\mathrm{Conj}(\mathrm{USp}(2g))$, which are equidistributed with respect to the image of Haar measure on $\mathrm{ST}(A)$.

Theorem (conditional!)

The classes in $\mathrm{Conj}(G)$ corresponding to the $\overline{L}_{A,\mathfrak{p}}(T)$ are equidistributed with respect to the image of Haar measure on some compact subgroup G of $\mathrm{USp}(2g)$.

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The case of elliptic curves

For $g = 1$, there are exactly three possibilities for $\mathrm{ST}(A)$.

- If A has complex multiplication defined over K, then $\mathrm{ST}(A) = \mathrm{SO}(2)$. Note that this case cannot occur if K is totally real.
- If A has complex multiplication not defined over K, then $\mathrm{ST}(A)$ is the normalizer of $\mathrm{SO}(2)$ in $\mathrm{USp}(2) = \mathrm{SU}(2)$. This group has 2 connected components; on the nonneutral component the trace is identically 0. (The primes that land there are the supersingular primes!)
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The Mumford-Tate group and the Sato-Tate group

Choose[3] an embedding $K \hookrightarrow \mathbb{C}$. Using any polarization on A, we may equip $V = H_1(A_{\mathbb{C}}^{\text{top}}, \mathbb{Q})$ with a symplectic pairing.

Also, $V_{\mathbb{R}} \cong H_1(A_{\mathbb{C}}^{\text{top}}, \mathbb{R})$ admits a complex structure coming from the complex uniformization of A. In particular, it admits an action of $\mathbb{C}^\times$.

The *Mumford-Tate group* of A is the minimal $\mathbb{Q}$-algebraic subgroup $\mathrm{MT}(A)$ of $\mathrm{Sp}(V)$ whose extension to $\mathbb{R}$ contains the $\mathbb{C}^\times$-action. In particular, it is a *connected* reductive algebraic group.

The neutral component of $\mathrm{ST}(A)$ is a maximal compact subgroup of $\mathrm{MT}(A)(\mathbb{C})$.

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Endomorphisms and the Sato-Tate group

Under favorable[4] conditions, the group $\mathrm{MT}(A)$ can also be interpreted as the maximal $\mathbb{Q}$-algebraic subgroup of $\mathrm{Sp}(V)$ which commutes with the action of $\mathrm{End}(A_{\overline{K}})$ on V.

In these cases, we may enlarge $\mathrm{MT}(A)$ to an *algebraic Sato-Tate group* $\mathrm{AST}(A)$ by considering elements which normalize $\mathrm{End}(A_{\overline{K}})$ via an element of $G_K = \mathrm{Gal}(\overline{K}/K)$. The full Sato-Tate group $\mathrm{ST}(A)$ is a maximal compact subgroup of $\mathrm{AST}(A)_{\mathbb{C}}$.

In particular, the component group of $\mathrm{ST}(A)$ is naturally identified with $\mathrm{Gal}(L/K)$ for some finite Galois extension L of K. In fact, L is the minimal field of definition of the endomorphisms of $A_{\overline{K}}$.

[4] This includes when $g \leq 3$. Otherwise, one must consider not just endomorphisms but also *absolute Hodge cycles* on A.

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Galois image and the Sato-Tate group

Pick a prime ℓ. Under favorable[5] conditions, the group $\mathrm{AST}(A)_{\mathbb{Q}_\ell}$ is the Zariski closure of the image of G_K acting on the ℓ-adic Tate module of A.

In these cases, each prime ideal $\mathfrak{p}$ of K at which A has good reduction gives rise to a conjugacy class in $\mathrm{ST}(A)$ by mapping the Frobenius class in G_K to $\mathrm{AST}(A)_{\mathbb{Q}_\ell}$, mapping further into $\mathrm{AST}(A)_{\mathbb{C}}$ via some embedding $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$, dividing by $q^{1/2}$, and semisimplifying.

Question: is there a good automorphic analogue of this construction? We are effectively looking for the smallest subgroup of $\mathrm{GSp}(2g)$ from which the given automorphic representation arises via base change.

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Endomorphism algebras and Sato-Tate groups

From now on, assume[6] $g = 2$.

Theorem

The group $\mathrm{ST}(A)$ *determines, and is uniquely determined by, the* $\mathbb{R}$*-algebra* $\mathrm{End}(A_{\overline{K}})_{\mathbb{R}}$ *together with its* G_K*-action. In particular, the connected subgroup of* $\mathrm{ST}(A)$ *determines, and is determined by,* $\mathrm{End}(A_{\overline{K}})_{\mathbb{R}}$*.*

$\mathrm{ST}(A)^\circ$	$\mathrm{End}(A_{\overline{K}})_{\mathbb{R}}$	How this group occurs
$\mathrm{USp}(4)$	$\mathbb{R}$	simple, no extra endomorphisms
$\mathrm{SU}(2) \times \mathrm{SU}(2)$	$\mathbb{R} \times \mathbb{R}$	simple RM *or* non-CM times non-CM
$\mathrm{SO}(2) \times \mathrm{SU}(2)$	$\mathbb{C} \times \mathbb{R}$	CM times non-CM
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[6] The case $g = 3$ is in principle tractable but involves hundreds (thousands?) of cases.

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Component groups

Theorem

Up to conjugation in USp(4)*, there are* 52 *possible groups* ST(A)*. Of these, exactly* 34 *occur over* $\mathbb{Q}$*; one more occurs over real quadratic fields.*

$\mathrm{ST}(A)^\circ$	Options for $\mathrm{ST}(A)/\mathrm{ST}(A)^\circ$ (* = realizable over $\mathbb{Q}$)
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$\mathrm{SU}(2)\times\mathrm{SU}(2)$	C_1^*, C_2^*
$\mathrm{SO}(2)\times\mathrm{SU}(2)$	C_1, C_2^*
$\mathrm{SO}(2)\times\mathrm{SO}(2)$	$C_1, C_2, C_2, C_4^*, D_2^*$
$\mathrm{SU}(2)$	$C_1^*, C_2^*, C_3^*, C_4^*, C_6^*, C_2^*, D_2^*, D_3^*, D_4^*, D_6^*$
$\mathrm{SO}(2)$	$C_1, C_2, C_3, C_4, C_6, D_2, D_3, D_4, D_6, A_4, S_4,$ $C_2, D_2^*, C_6, C_4\times C_2^*, C_6\times C_2^*, D_2\times C_2^*, D_6^*,$ $D_4\times C_2^*, D_6\times C_2^*, A_4\times C_2^*, S_4\times C_2^*,$ $C_2^*, C_4, C_6^*, D_2^*, D_4^*, D_6^*, D_3^*, D_4^*, D_6^*, S_4^*$

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Moment sequences

Theorem

The 52 possible groups ST(A) are distinguished by the moments

$$\mathbb{E}(a_1^2), \mathbb{E}(a_1^4), \mathbb{E}(a_1^6), \mathbb{E}(a_1^8), \mathbb{E}(a_2), \mathbb{E}(a_2^2), \mathbb{E}(a_2^3), \mathbb{E}(a_2^4).$$

In practice, fewer moments are needed. For instance, the group USp(4) has $\mathbb{E}(a_1^4) = 3$ and all other groups have $\mathbb{E}(a_1^4) \geq 4$. This distinction can be detected in practice using only a few hundred primes!

Especially for Jacobians of genus 2 curves, it is relatively efficient to compute normalized L-polynomials; these can then be used to detect ST(A) and even more refined data.

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A word on unconditional results

The classification of Sato-Tate groups for abelian surfaces is unconditional, in part because the Mumford-Tate conjecture is known for abelian surfaces.

The equidistribution is unconditional in all cases where $\mathrm{ST}(A)^\circ$ is a torus (in all dimensions). This reduces to results of Hecke.

For abelian surfaces with $\mathrm{ST}(A)^\circ = \mathrm{SU}(2), \mathrm{SO}(2) \times \mathrm{SU}(2), \mathrm{SU}(2) \times \mathrm{SU}(2)$, equidistribution has been shown by Johansson provided that K and a certain quadratic extension are both totally real. This uses hard potential automorphy theorems of Harris, Taylor, etc.

For abelian surfaces with $\mathrm{ST}(A) = \mathrm{USp}(4)$, equidistribution is known in no cases. One needs potential automorphy for L-functions associated to *all* representations of $\mathrm{USp}(4)$, not just symmetric powers.

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