# Frobenius slope filtrations and Crew's conjecture

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August 20, 2003

These are notes from a lecture given at the workshop "Current trends in arithmetic geometry and number theory," held at the Banff International Research Station, August 17–21, 2003. The lecture was the last in a series of instructional lectures concerning $p$-adic Hodge theory (Brian Conrad), $(\phi, \Gamma)$-modules (Adrian Iovita, Nathalie Wach, Pierre Colmez), and the connection to $p$-adic differential equations (Laurent Berger); a motivating goal was to describe the proof of Fontaine's conjecture that "de Rham representations are potentially semistable." This talk was accordingly given in that context, so not everything may make sense in isolation.

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# 0 Context

Earlier in the lecture series, we discussed the conjecture of Fontaine that de Rham representations (of the Galois group of a local field of mixed characteristics) are potentially semistable. Berger's construction reduced this problem to a question about $p$-adic differential equations, known variously as "Crew's conjecture" (as in my title), more properly the Crew-Tsuzuki conjecture (as in Laurent's talk), or the $p$-adic local monodromy theorem (now that it is not conjectural!). Namely, every "$p$-adic differential equation with Frobenius structure" should be quasi-unipotent; i.e., after a finite base extension, it should admit a basis which is nilpotent for the connection.

There are two basic strategies available for proving the theorem. One is to use strong properties of $p$-adic differential equations and then fix on the Frobenius structure at the end: this is the strategy used independently by André and Mebkhout, but they share the use of work of Christol and Mebkhout (that in turn uses ideas of Robba and ultimately Dwork). I won't speak further on this, as it would take us pretty far afield from the focus of the conference.

The other strategy, used in my paper "A $p$-adic local monodromy theorem" (to appear in *Annals of Math.*, but it's on the arXiv already), is to use strong properties of Frobenius structures, in the absence of any connection, and then fix on the connection at the end. This uses ideas of de Jong, Tsuzuki, Crew and ultimately Dwork; it's also very much in the spirit of the discussion of $(\phi, \Gamma)$ modules and "big rings" of the previous lectures.

To put the properties of Frobenius structures over the Robba ring in context, we recall a more classical structure theorem for Frobenius structures, the Dieudonné-Manin classification.

# 1 Dieudonné-Manin decompositions

## 1.1 $\phi$-modules and the Dieudonné-Manin classification

Notation is not yet as in the other talks, since I'm discussing an auxiliary situation. Also, see Katz's article "Slope filtrations of $F$-crystals" for a comprehensive reference on this topic.

Let $R$ be a complete discrete valuation ring of mixed characteristics $(0, p)$ equipped with an endomorphism $\phi : R \to R$ lifting the $p$-power Frobenius on the residue field. For instance, $R$ could be the ring of Witt vectors of a field of characteristic $p$. (Actually, $\phi$ could lift the $q$-power Frobenius for some power $q$ of $p$ if you prefer, but for simplicity let me avoid that level of generality.) Let $L$ be the fraction field of $R$. We normalize the valuation $v$ on $R$ so that $v : L^* \to \mathbb{Z}$ is surjective.

A *$\phi$-module* $M$ (also a called an $F$-module) over $L$ is a finite dimensional $L$-vector space $M$ equipped with a semilinear $\phi$-action (that is, $\phi(cm) = \phi(c)\phi(m)$ for $c \in L$ and $m \in M$) which is nondegenerate in the sense that the image of $\phi$ spans $M$. You can specify such a thing by giving the action on a basis, in which case the defining matrix must be invertible. (Another way to say it: the induced linear map $\phi^* M \to M$ is an isomorphism.)

In linear algebra, the structure of a vector space equipped with a linear transformation is most simple over an algebraically closed field. Similarly, for $R$ with algebraically closed residue field, a structure theorem attributed to Dieudonné and Manin classifies these objects completely. First let's introduce our basic objects. For $\lambda \in L^*$ and $n$ a positive integer, let $M_{n,\lambda}$ be the $\phi$-module of rank $n$ given by

$$\phi \begin{pmatrix} a_1 & a_2 & \vdots & a_n \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \lambda \\ 1 & \cdots & 0 & 0 & \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 & \end{pmatrix} \begin{pmatrix} \phi(a_1) & \phi(a_2) & \vdots & \phi(a_n) \end{pmatrix}.$$

It can be shown that $M_{n,\lambda}$ is irreducible if and only if $\gcd(v(\lambda), n) = 1$.

**Theorem 1 (Dieudonné-Manin classification).** *Assume that $R$ has algebraically closed residue field (otherwise all of the following statements are false).*

*(a) Any irreducible $\phi$-module over $L$ is isomorphic to $M_{n,\lambda}$ for some $n$ and $\lambda$ with $\gcd(n, v(\lambda)) = 1$.*

*(b) The $\phi$-modules $M_{n,\lambda}$ and $M_{n',\lambda'}$ are isomorphic if and only if $n = n'$ and $v(\lambda)/n = v(\lambda')/n'$. The latter quantity is called the* slope *of $M_{n,\lambda}$.*

*(c) The category of $\phi$-modules over $L$ is semisimple. Hence each $\phi$-module $M$ over $L$ is isomorphic to a direct sum of $M_{n,\lambda}$'s. If you prefer, each $M$ admits a* canonical *decomposition as a direct sum of isotypical modules (each of which is noncanonically a direct sum of copies of a single irreducible $\phi$-module), called the* slope decomposition.

An example where (c) fails without algebraically closed residue field:

$$\phi \begin{pmatrix} a & b \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi(a) & \phi(b) \end{pmatrix}$$

We get a short exact sequence by quotienting by the submodule generated by $(1, 0)$. To split it, we must find $(y, 1)$ such that $\phi(y) + x = y$, which may not exist.

## 1.2 Newton polygons

When $R$ has algebraically closed residue field, we define the multiset of *slopes* of a $\phi$-module $M$ to include $v_p(\lambda)/n$ with multiplicity $nr$ if $M$ contains $r$ direct summands isomorphic to $M_{n,\lambda}$. (So the number of slopes equals the dimension of $M$.)

We can form a Newton polygon out of these: start at $(0, 0)$, draw a segment of horizontal width 1 to the right whose slope is the smallest slope of $M$, then draw a segment of horizontal width 1 to the right of that whose slope is the second smallest slope of $M$, etc. We call this the *generic Newton polygon* of $M$; the term "generic" will be justified later.

The $M_{n,\lambda}$ are a bit like "eigenspaces". However, as Brian noted in his talk, it is *not* true that one can read off the slopes of $M$ as the valuations of the eigenvalues of the matrix via which $\phi$ acts on an *arbitrary* basis of $M$. This only works for certain "good" bases. For example, if $\mathbf{v}, \phi\mathbf{v}, \dots, \phi^{\dim(M)-1}\mathbf{v}$ are linearly independent (that is, $\mathbf{v}$ is a *cyclic vector*), then they form a good basis; you can even use cyclic vectors to prove the existence of Dieudonné-Manin decompositions (à la de Jong).

The slopes do act like valuations of eigenvalues in various formal ways, though. For example, the Newton polygon has integral vertices (if the valuation is normalized), and the slopes of a tensor product are the pairwise sums of the slopes of the factors.

### 1.3 A slope filtration

What happens if the residue field of $R$ is not algebraically closed (e.g., $k((t))$)? One can still talk about slopes and the generic Newton polygon, by passing up to the completed maximal unramified extension of $R$. (Actually, that's only if $R$ has perfect residue field. Otherwise, one must first "perfectify" by taking the completed direct limit of the system $R \to R \to \cdots$, where the maps are all $\phi$.) But what can we say without changing base?

First of all, if the residue field of $R$ is perfect, one still has the slope decomposition of $M$ (by Galois descent), although it's not an isotypical decomposition per se: the slope pieces no longer decompose as the sum of $M_{n,\lambda}$'s.

For $R$ with arbitrary residue field, one need not even have a slope decomposition: for instance, if

$$\phi \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & p \end{pmatrix} \begin{pmatrix} \phi(a) \\ \phi(b) \end{pmatrix}$$

for $x \in R$, one can split the obvious short exact sequence if and only if $\phi(y) + x = py$; in particular, $x$ must reduce to a $p$-power in the residue field. On the other hand, if the roles of 1 and $p$ were reversed above, one would always get a splitting, since the equation $p\phi(y) + x = y$ is always solvable!

The general fact underlying this observation is that $M$ always has a canonical *(ascending) generic slope filtration* in which for each $i$, $\mathrm{Gr}^i M = \mathrm{Fil}^i M / \mathrm{Fil}^{<i} M$ has all slopes equal to $i$. (Here $\mathrm{Fil}^{<i} M = \cup_{j<i} \mathrm{Fil}^j M$.)

## 2 A slope filtration over the Robba ring

We now describe an analogue of the ascending generic slope filtration for $\phi$-modules over the Robba ring, and its relevance to $p$-adic monodromy. It comes from an analogue of the Dieudonné-Manin decomposition, but more on that later.

Notation is now as in the previous talks (i.e., as in Berger's thesis), though the theorem is actually true in more generality (e.g., $k$ need not be perfect). That is, $K$ is a complete discretely valued field of characteristic 0 with perfect residue field $k$ of characteristic $p > 0$, and $K_0$ is the maximal unramified subextension of $K$. The ring $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ is the Robba ring over $K_0$, which can be viewed as containing Laurent series in some parameter $T$ (a/k/a $\pi_K$) which converge on some annulus of the form $\eta < |T| < 1$; its subring $\mathbf{B}^\dagger_K$ consists of series with bounded coefficients, and is a complete discretely valued field with residue field $k((t))$. The rings $\mathbf{B}^\dagger_{\mathrm{rig},K}$ and $\mathbf{B}^\dagger_K$ are equipped with a Frobenius endomorphism $\phi$, which acts as the $p$-power Frobenius on the residue field of $\mathbf{B}^\dagger_K$ (and preserves the valuation on $\mathbf{B}^\dagger_K$).

## 2.1 The special slope filtration

Now let $M$ be a $\phi$-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$, by which I again mean a finite free $\mathbf{B}^\dagger_{\mathrm{rig},K}$-module equipped with a nondegenerate semilinear action of $\phi$. (Beware: "nondegenerate" means that a defining matrix has to be invertible, not just nonsingular! So its determinant must be a unit in $\mathbf{B}^\dagger_{\mathrm{rig},K}$, i.e., a nonzero element of $\mathbf{B}^\dagger_K$.) Since $\mathbf{B}^\dagger_{\mathrm{rig},K}$ is not a discrete valuation ring, there is no *a priori* notion of the "slopes" of $M$. That notion is provided by the following theorem.

**Theorem 2.** *Let $M$ be a $\phi$-module over $\mathbf{B}^\dagger_{\mathrm{rig},K}$. Then $M$ admits a canonical increasing filtration (by $\phi$-submodules) such that for each $i$, $\mathrm{Gr}^i M = \mathrm{Fil}^i M/\mathrm{Fil}^{<i} M$ is (free and) isomorphic to a $\phi$-module over $\mathbf{B}^\dagger_K$ whose slopes are all equal to $i$.*

We call this filtration the *(ascending) special slope filtration*; correspondingly, whenever $\mathrm{Gr}^i M = \mathrm{Fil}^i M/\mathrm{Fil}^{<i} M$ is nonzero, we call $i$ a *slope* of $M$ with multiplicity $\mathrm{rank}(\mathrm{Gr}^i M)$. Again, these form a Newton polygon, called the *special Newton polygon*.

The adjectives "generic" and "special" are justified by the case when $M$ itself is defined over $\mathbf{B}^\dagger_K$; then $M$ has two Newton polygons, one generic (obtained by passing to the completion $\mathbf{B}_K$) and one special (obtained by passing to $\mathbf{B}^\dagger_{\mathrm{rig},K}$ and using the slope filtration described above), related as follows.

**Proposition 3.** *If $M$ is a $\phi$-module over $\mathbf{B}^\dagger_K$, its special Newton polygon lies on or above its generic Newton polygon, and both have the same endpoints.*

To justify the terminology further: if $M$ comes from the $i$-th cohomology of a family of smooth proper varieties, or even a semistable family, then the generic/special Newton polygon of $M$ is the Newton polygon of the $i$-th cohomology of the generic/special fibre, and the inequality is a consequence of Grothendieck's specialization theorem.

For example, the $\phi$-module given by

$$\phi \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & p \\ 1 & T \end{pmatrix} \begin{pmatrix} \phi(a) \\ \phi(b) \end{pmatrix}$$

has generic Newton slopes 0 and 1, and special Newton slopes 1/2 and 1/2. This is a special case of the following general phenomenon: if $M$ is defined over the subring of $\mathbf{B}^\dagger_{\mathrm{rig},K}$ consisting only of nonnegative powers of $T$, then the special Newton polygon is the Newton polygon of the "specialization" $M/TM$ (by "Dwork's trick").

## 2.2 Slope filtrations and local monodromy

Now say $M$ is a $\phi$-module with connection. Why does the existence of a slope filtration imply the $p$-adic local monodromy theorem? It immediately reduces the question (existence of a unipotent filtration after a finite base extension) for $M$ to the same question for its graded pieces, which are defined over $\mathbf{B}_K^\dagger$ and have one slope each. By twisting, one can reduce to the case where the Frobenius has slope zero; in this case, $M$ is isomorphic to a $\phi$-module with connection over $\mathbf{A}_K^\dagger$ (the ring of integers of $\mathbf{B}_K^\dagger$). In this case, we say $M$ is *étale* (or sometimes *unit-root*).

That case of the monodromy theorem is much easier than the general case, and is due to Tsuzuki. (In fact, Laurent mentioned it in his discussion of $\mathbb{C}_p$-admissible representations.) In this case, one shows that if $\phi$ acts on some basis of $M$ via a matrix $\Phi$ with $v_p((\Phi - I)_{ij}) < v_p(p^{1/(p-1)})$, then $M$ has a basis of horizontal sections (not just a unipotent basis). You get to this point by trivializing $M$ modulo successive powers of the maximal ideal, which consists of adjoining roots of various polynomials to $k((t))$. (In particular, you can "read off" the base extension needed to get the unipotent basis. It's not clear how to do that for a general $\phi$-module, which is what makes Crew's conjecture hard to prove!)

# 3 Construction of the special slope filtration

The existence of the special slope filtration is the main theorem of my *long* preprint "A $p$-adic local monodromy theorem", so a mere sketch will have to suffice.

## 3.1 Another Dieudonné-Manin decomposition

To produce the slope filtration, we first need an analogue of the Dieudonné-Manin decomposition over the Robba ring, or rather, over some large overring. What then is the analogue over $\mathbf{B}_{\mathrm{rig},K}^\dagger$ of replacing a complete discrete valuation ring with the completion of its maximal unramified extension?

In fact, Laurent constructed the right overring in his talk: it is the ring he denotes $\tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$ (the "one ring to rule them all"). Facts to recall about this ring:

- It contains the maximal unramified extension of $\mathbf{B}_K^\dagger$.
- It is complete with respect to a limit-of-Fréchet topology, just like $\mathbf{B}_{\mathrm{rig},K}^\dagger$. (This makes it analogous to the completed maximal unramified extension of a complete DVR.)
- The units in $\tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$, together with zero, form a complete discretely valued field (just as the units of $\mathbf{B}_{\mathrm{rig},K}^\dagger$ plus zero form the field $\mathbf{B}_K^\dagger$), so we can apply $v_p$ to them sensibly.

Now for the classification, which *mostly* resembles the Dieudonné-Manin classification.

**Theorem 4.** *(a) Any irreducible $\phi$-module over $\tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger$ is isomorphic to $M_{n,\lambda}$ for some positive integer $n$ and some unit $\lambda \in (\tilde{\mathbf{B}}_{\mathrm{rig}}^\dagger)^*$ whose valuation is coprime to $n$.*

*(b) The $\phi$-modules $M_{n,\lambda}$ and $M_{n',\lambda'}$ are isomorphic if and only if $n = n'$ and $v_p(\lambda)/n = v_p(\lambda')/n'$. (The valuations make sense because $\lambda$ and $\lambda'$ are units.)*

*(c) Every $\phi$-module over $\tilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ is a direct sum of $M_{n,\lambda}$'s.*

However, the category is not semisimple: not every short exact sequence of $\phi$-modules over $\tilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ splits! For example, in case $\phi(T) = T^p$, one can show that the obvious short exact sequence containing

$$\phi \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} p & T^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi(a) \\ \phi(b) \end{pmatrix}$$

does not split. More on this momentarily.

Aside: an equal characteristic analogue of this result has been established by Hartl and Pink, with a very similar proof. (They are classifying vector bundles on a punctured open unit disc over the equal-characteristic analogue of $\mathbb{C}_p$, equipped with a Frobenius structure.)

## 3.2 Steps of the proof

Some of the key steps of the construction:

- Show that like the Robba ring $\mathbf{B}^{\dagger}_{\mathrm{rig},K}$ (theorem of Lazard), $\tilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}}$ is a Bézout ring: every finitely generated ideal is principal. One actually shows this by doing it for $\tilde{\mathbb{B}}^{\dagger,r}_{\mathrm{rig},K}$ for each $r$; the proof is analogous to Lazard's proof in the Robba case. (Such rings are "morally" principal ideal domains as far as finite operations are concerned.) In particular, projective submodules of a $\phi$-module are free (big help!).

- Show that any $\phi$-module admits a maximal linearly independent set of "eigenvectors". From this one easily deduces that $M$ admits a "backwards" filtration, a filtration whose successive quotients are $M_{n,\lambda}$'s, but whose slopes *decrease* as you go up. Note what Newton polygon you get.

- Show that if this filtration does not split, you can find a new one whose Newton polygon is *strictly above* the one given by the short exact sequence (and has the same endpoint). It suffices to do the case of a two-step filtration (a short exact sequence). In the above example, one gets a new basis where

$$\phi \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi(c) \\ \phi(d) \end{pmatrix}.$$

- Deduce that eventually one gets a filtration of $M$ that splits (since the Newton polygons of the filtrations have fixed endpoints, integer vertices, and keep rising, which they can't do forever).

### 3.3 Descent

One does *not* get an isotypical decomposition in this setting. All one gets is a canonical increasing filtration $\mathrm{Fil}^i M$ by free $\phi$-submodules. This certainly descends to the Galois-invariant subring of $\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$; one must somehow show that as in the "generic" case, this filtration actually goes down "all the way" to $\mathbf{B}^\dagger_{\mathrm{rig},K}$.

This is quite technical, and I won't say anything detailed about it here. The short version: using the special slope filtration, one is guided to a sequence of changes of basis over $\mathbf{B}^\dagger_{\mathrm{rig},K}$ which have the net effect of producing a $\phi$-module over $\mathbf{B}^\dagger_K$ whose generic and special Newton polygons coincide. Then you show that the ascending generic slope filtration over the completion of $\mathbf{B}^\dagger_K$ actually descends to $\mathbf{B}^\dagger_K$. (One interpolates from generic to special using a "descending generic slope filtration" constructed by de Jong, much as the Charbonnier-Colmez construction "interpolates" between incompatible period rings.)

# 4 Other applications

Note: this section did not occur in the talk itself for lack of time.

We already know one application of these results, namely Fontaine's conjecture. There are several others.

Berthelot's theory of rigid cohomology, a generalization of crystalline cohomology that behaves well on nonsmooth, nonproper varieties, depends heavily on the $p$-adic local monodromy theorem for various finiteness properties; many of these have now been worked out by Crew and myself. In fact, various arguments parallel both the "Frobenius" and "differential" approaches to the $p$-adic local monodromy theorem, so it is valuable to have both!

Another application of the slope filtration itself is a variant of Crew's conjecture due to Yves André and Lucia di Vizio; they consider "$q$-deformations of $p$-adic differential equations". The analogue of the differential structure changes significantly, but the analogue of Frobenius structure remains unchanged, so one needs only analogize Tsuzuki's étale result, which is relatively easy to do. This theory may be relevant to deformation of rigid cohomology, or maybe even of Galois representations.