### A brief (pre)history of perfectoid spaces

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#### Short answer: see B. Bhatt, Notices of the AMS, October 2014.

Slightly longer answer: perfectoid spaces are a new class of objects in arithmetic algebraic geometry which provide a strong link between geometry in characteristic 0 and characteristic p.

This talk will not attempt to define perfectoid spaces. Instead, I will trace the genesis of these ideas through a series of key developments in arithmetic geometry over the past 50 years. (For the rest of the history, see Scholze, Scholze-Weinstein, Kedlaya-Liu, Fargues-Fontaine, Bhatt-Scholze, Bhatt-Morrow-Scholze, ...; also Scholze's talk at CDM 2015 next month.) Short answer: see B. Bhatt, Notices of the AMS, October 2014.

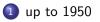
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Consider the following map from the complex plane to itself:

$$z\mapsto z^2.$$

For most points  $z \in \mathbb{C}$ , the inverse image of z consists of two distinct points. In fact, one can even find a neighborhood U of z whose inverse image consists of two disjoint copies of U.

However, for z = 0, the inverse image is a single point, and the local structure of neighborhoods of these points and their inverse images is a bit more complicated.

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# Ramification in algebra(ic geometry)

Using formal power series, we can give an algebraic reinterpretation of the previous discussion:

$$\frac{\mathbb{Q}[\![z-t]\!][x]}{(x^2-z)} \cong \begin{cases} \mathbb{Q}[\![x-\sqrt{t}]\!] \oplus \mathbb{Q}[\![x+\sqrt{t}]\!] & (t\neq 0, t=\Box) \\ \mathbb{Q}(\sqrt{t})[\![z-t]\!] & (t\neq 0, t\neq \Box) \\ \mathbb{Q}[\![x]\!] & (t=0). \end{cases}$$

The key point: z - t vanishes to order 1 in each factor when t = 1, but to order 2 when t = 0. Note that in  $\mathbb{Q}[x - \sqrt{t}]$  we have

$$\frac{1}{x + \sqrt{t}} = \frac{1/(2\sqrt{t})}{1 + (x - \sqrt{t})/(2\sqrt{t})}$$
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## Ramification in number theory

Let *K* be a number field, i.e., a field which is a finite extension of  $\mathbb{Q}$ . The integral closure of  $\mathbb{Z}$  in *K* is denoted  $\mathfrak{o}_{K}$ . E.g., if

$$\mathcal{K} = \mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}, \text{ then } \mathfrak{o}_{\mathcal{K}} = \mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}.$$

The analogue of a ring of formal power series (resp. formal Laurent series) is the completion of  $\mathfrak{o}_{\mathcal{K}}$  with respect to a prime ideal  $\mathfrak{p}$ , denoted  $\mathfrak{o}_{\mathcal{K}_{\mathfrak{p}}}$  (resp. the fraction field of this completion, denoted  $\mathcal{K}_{\mathfrak{p}}$ ). Elements of  $\mathfrak{o}_{\mathcal{K}}$  can be viewed as coherent sequences of residue classes modulo  $\mathfrak{p}$ , modulo  $\mathfrak{p}^2$ , etc.

E.g., for  $K = \mathbb{Q}$ ,  $\mathfrak{p} = (p)$ ,  $\mathfrak{o}_{K_{\mathfrak{p}}} = \mathbb{Z}_p$  is Hensel's ring of *p*-adic numbers and  $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$ . Formally, elements of  $\mathbb{Z}_p$  may be viewed as "infinite base *p* numerals"

$$a_0 + a_1 p + a_2 p^2 + \cdots, \qquad a_i \in \{0, \dots, p-1\}.$$

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### What does $\mathbb{Z}_p[i]$ (or more precisely $\mathbb{Z}_p[i]/(i^2+1)$ ) look like?

- If p ≡ 1 (mod 4), then it splits as two copies of Z<sub>p</sub>. For example, if p = 5, then 2 + i is divisible by 5 in one copy of Z<sub>5</sub> but is invertible in the other.
- If p ≡ 3 (mod 4), then Z<sub>p</sub>[i] is an integral domain, and every nonzero ideal is a power of (p).
- If p = 2, then Z<sub>p</sub>[i] is again an integral domain, but now the ideal (1 + i) is not a power of (2).

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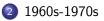
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For  $x \in L_q$ , the *trace* of x is the trace of multiplication-by-x as a  $K_p$ -linear transformation on  $L_q$ . (It is also the sum of the Galois conjugates of x.)

#### Lemma

The trace map takes  $o_{L_q}$  into  $o_{K_p}$ , and is surjective if and only if q is not a ramified prime.

For example, take  $K = \mathbb{Q}$ ,  $L = \mathbb{Q}(i)$ ,  $\mathfrak{p} = (2)$ ,  $\mathfrak{q} = (1 + i)$ . Then

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Tate discovered that ramification "almost disappears" if one makes certain *infinite* extensions of number fields.

E.g., for any prime p, let  $\mathbb{Z}_p[\zeta_{p^{\infty}}]$  (resp.  $\mathbb{Q}_p(\zeta_{p^{\infty}})$ ) be the ring obtained from  $\mathbb{Z}_p$  (resp.  $\mathbb{Q}_p$ ) by adjoining primitive  $p^n$ -th roots of unity  $\zeta_{p^n}$  for all n. (Note that  $\mathbb{Z}_p[\zeta_{p^{\infty}}]$  is the integral closure of  $\mathbb{Z}_p$  in  $\mathbb{Q}_p(\zeta_{p^{\infty}})$ .)

#### Theorem (example of a theorem of Tate)

Let F be a finite extension of  $\mathbb{Q}_p(\zeta_{p^{\infty}})$ . Let  $\mathfrak{o}_F$  be the integral closure of  $\mathbb{Z}_p$  in F. Then Trace :  $\mathfrak{o}_F \to \mathbb{Q}_p(\zeta_{p^{\infty}})$  has image in  $\mathbb{Z}_p[\zeta_{p^{\infty}}]$  and is almost surjective: its image contains the maximal ideal of  $\mathbb{Z}_p[\zeta_{p^{\infty}}]$ .

That maximal ideal is generated by  $1-\zeta_{p^n}$  for  $n=1,2,\ldots$ , and

$$(1-\zeta_p)^{p-1}=p\times(\operatorname{unit}),\qquad (1-\zeta_{p^{n+1}})^p=(1-\zeta_{p^n})\times(\operatorname{unit}).$$

So any element "divisible by  $p^{\epsilon}$  for some  $\epsilon > 0$ " is a trace.

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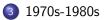
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#### 2 1960s-1970s









# The field of norms equivalence

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#### Theorem (example of a theorem of Fontaine-Wintenberger)

There is an explicit isomorphism between the absolute Galois groups of  $\mathbb{Q}_p(\zeta_{p^{\infty}})$  and  $\mathbb{F}_p((\pi))$ .

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## Artin-Schreier theory

One consequence of the previous theorem is that one can use special tools from characteristic p to study Galois theory in characteristic 0. One of the most powerful of these is the Artin-Schreier construction.

For any ring R of characteristic p, the map  $x \mapsto x^p$  on R is a ring homomorphism, called *Frobenius* and denoted  $\varphi$ .

For any *R*-module *M*, we can twist the action of *R* on *M* to define a new *R*-module  $M \otimes_{\varphi} R$ :

$$rm \otimes 1 = m \otimes \varphi(r).$$

We define a  $\varphi$ -module over R to be a finite projective R-module M equipped with an R-linear isomorphism  $F : M \otimes_{\varphi} R \cong M$ . The map  $\Phi : M \to M$  given by  $\Phi(m) = F(m \otimes 1)$  is  $\varphi$ -semilinear:

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# $\varphi$ -modules and Galois representations

Let F be a field of characteristic p with absolute Galois group  $G_F$ . The group  $G_F$  is a compact profinite topological group, and the Galois groups of finite separable extensions of E occur as open subgroups.

Theorem (Katz)

The functors

$$V \mapsto (V \otimes_{\mathbb{F}_p} F^{sep})^{\mathcal{G}_F}, \qquad D \mapsto (D \otimes_F F^{sep})^{\varphi}$$

define equivalences of categories

 $\begin{cases} continuous^a \ G_F\text{-representations} \\ on \ finite \ \mathbb{F}_p\text{-vector spaces } V \end{cases} \leftrightarrow \{\varphi\text{-modules } D \ over \ F \}.$ 

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(And something similar for representations on modules over  $\mathbb{Z}_p$  or  $\mathbb{Q}_p$ .)

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A brief (pre)history of perfectoid spaces

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# Related developments

In related work, Fontaine formalized an observation of Tate and conjectured the form of a *comparison isomorphism* between étale and de Rham cohomology for smooth proper algebraic varieties over finite extensions of  $\mathbb{Q}_p$ .

Fontaine also introduced the concept of a  $(\varphi, \Gamma)$ -module, which allows the theorem of Katz to be applied to finite extensions of  $\mathbb{Q}_p$  (whose absolute Galois groups do not directly appear in characteristic p).

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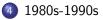
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## 5 2000s



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# What is really going on here?

In the mid-2000s, I tried to understand the proofs of the theorems of Fontaine-Wintenberger and Faltings. Failing to do so, I decided to try to come up with my own proofs. Some progress was reported at ICM 2010; subsequent progress includes joint work with Ruochuan Liu.

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# A construction of Fontaine

For any ring R, the ring R/(p) is of characteristic p, so it has a Frobenius map  $\varphi$ . We can then define

$$R'=\varprojlim_{\varphi}R/(p);$$

that is, R' consists of sequences  $(\ldots, x_1, x_0)$  in R/(p) such that  $x_{n+1}^p = x_n$ . The ring R' is again of characteristic p, but now  $\varphi$  is bijective: its inverse is the map  $(\ldots, x_1, x_0) \to (\ldots, x_2, x_1)$ .

## Theorem (Fontaine)

If R is the integral closure of  $\mathbb{Z}_p$  in an algebraic closure of  $\mathbb{Q}_p$ , then R' is a valuation ring in an algebraically closed field.

But what about smaller R? For example, if  $R = \mathbb{Z}_p[\zeta_{p^{\infty}}]$ , then R' is the  $\pi$ -adic completion of  $\mathbb{F}_p[\pi, \pi^{1/p}, \dots]$ .

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# The Fontaine-Wintenberger equivalence revisited

The following is in some sense a *maximal* generalization of the Fontaine-Wintenberger theorem.

## Theorem (K-Liu, Scholze)

Let K be the completion of an algebraic extension of  $\mathbb{Q}_p$ . Suppose that: (a) the valuation on K is not discrete (so  $[K : \mathbb{Q}_p] = \infty$ ); (b) the map  $\varphi : \mathfrak{o}_K/(p) \to \mathfrak{o}_K/(p)$  is surjective. Then the ring  $(\mathfrak{o}_K)'$  is a valuation ring in a field K' of characteristic p, and there is a canonical isomorphism of Galois groups  $G_K \cong G_{K'}$ .

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## More on the Fontaine-Wintenberger equivalence

For any ring S of characteristic p on which  $\varphi$  is bijective, there is a unique p-adically separated and complete ring W(S) such that  $W(S)/(p) \cong S$ . It can be built explicitly using *Witt vectors*. For instance, if  $S = \mathbb{F}_p$ , then  $W(S) \cong \mathbb{Z}_p$ , but with a more exotic description.

Fontaine showed that for R, R' as before, there is an explicit map  $\theta : W(R') \to R$ . The conditions of the theorem imply that  $\theta : W(\mathfrak{o}_{K'}) \to \mathfrak{o}_K$  is surjective. One then shows that for every finite extension L' of K',

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## The Fontaine construction for rings

Let *R* be a ring which is *p*-torsion-free and integrally closed in R[1/p] (e.g., we rule out  $\mathbb{Z}_p[px, x^2]$  because *x* is missing). Suppose also that:

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Again, the map  $W(R') \to R$  is surjective. Let  $\pi \in R'$  be any element of the form  $(\ldots, x, p)$ ; we will then compare R[1/p] and  $R'[1/\pi]$  in the same way that we compared K and K' in the previous theorem.

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Let R, R' be as on the previous slide.

- (a) If R is p-adically separated and complete<sup>a</sup>, then there is an explicit equivalence of categories of finite étale algebras over R[1/p] and  $R'[1/\pi]$ .
- (b) Let S be the integral closure of R in some finite étale R[1/p]-algebra. Then Trace :  $S \rightarrow R$  is almost surjective.

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Again, (b) follows from (a) using the surjection  $\theta : W(R') \to R'$  and the map  $\varphi^{-1}$  on  $R'[1/\pi]$ . To prove (a), one reduces to the preceding theorem using a glueing argument in nonarchimedean analytic geometry.

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