A brief (pre)history of perfectoid spaces

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WHAT IS... a perfectoid space?

Short answer: see B. Bhatt, *Notices of the AMS*, October 2014.

Slightly longer answer: perfectoid spaces are a new class of objects in arithmetic algebraic geometry which provide a strong link between geometry in characteristic 0 and characteristic p.

This talk will not attempt to define perfectoid spaces. Instead, I will trace the genesis of these ideas through a series of key developments in arithmetic geometry over the past 50 years. (For the rest of the history, see Scholze, Scholze-Weinstein, Kedlaya-Liu, Fargues-Fontaine, Bhatt-Scholze, Bhatt-Morrow-Scholze, ...; also Scholze's talk at CDM 2015 next month.)

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Ramification in geometry

Consider the following map from the complex plane to itself:

$$z \mapsto z^2.$$

For most points $z \in \mathbb{C}$, the inverse image of z consists of two distinct points. In fact, one can even find a neighborhood U of z whose inverse image consists of two disjoint copies of U.

However, for $z = 0$, the inverse image is a single point, and the local structure of neighborhoods of these points and their inverse images is a bit more complicated.

If we replace $\mathbb{C}$ with a field which is not algebraically closed, like $\mathbb{Q}$, then even when $t \neq 0$ the two points can fail to separate; but this will not bother us too much.

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Ramification in algebra(ic geometry)

Using formal power series, we can give an algebraic reinterpretation of the previous discussion:

$$\frac{\mathbb{Q}[\![z-t]\!][x]}{(x^2-z)} \cong \begin{cases} \mathbb{Q}[\![x-\sqrt{t}]\!] \oplus \mathbb{Q}[\![x+\sqrt{t}]\!] & (t \neq 0, t = \square) \\ \mathbb{Q}(\sqrt{t})[\![z-t]\!] & (t \neq 0, t \neq \square) \\ \mathbb{Q}[\![x]\!] & (t = 0). \end{cases}$$

The key point: $z - t$ vanishes to order 1 in each factor when $t = 1$, but to order 2 when $t = 0$. Note that in $\mathbb{Q}[\![x-\sqrt{t}]\!]$ we have

$$\begin{aligned} \frac{1}{x+\sqrt{t}} &= \frac{1/(2\sqrt{t})}{1+(x-\sqrt{t})/(2\sqrt{t})} \\ &= \frac{1}{2\sqrt{t}} - \frac{x-\sqrt{t}}{(2\sqrt{t})^2} + \frac{(x-\sqrt{t})^2}{(2\sqrt{t})^3} - \cdots . \end{aligned}$$

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Ramification in number theory

Let K be a number field, i.e., a field which is a finite extension of $\mathbb{Q}$. The integral closure of $\mathbb{Z}$ in K is denoted $\mathfrak{o}_K$. E.g., if

$$K = \mathbb{Q}(i) = \{a + bi : a, b \in \mathbb{Q}\}, \text{ then } \mathfrak{o}_K = \mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}.$$

The analogue of a ring of formal power series (resp. formal Laurent series) is the completion of $\mathfrak{o}_K$ with respect to a prime ideal $\mathfrak{p}$, denoted $\mathfrak{o}_{K_\mathfrak{p}}$ (resp. the fraction field of this completion, denoted $K_\mathfrak{p}$). Elements of $\mathfrak{o}_K$ can be viewed as coherent sequences of residue classes modulo $\mathfrak{p}$, modulo $\mathfrak{p}^2$, etc.

E.g., for $K = \mathbb{Q}$, $\mathfrak{p} = (p)$, $\mathfrak{o}_{K_\mathfrak{p}} = \mathbb{Z}_p$ is Hensel's ring of *p-adic numbers* and $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$. Formally, elements of $\mathbb{Z}_p$ may be viewed as "infinite base p numerals"

$$a_0 + a_1 p + a_2 p^2 + \cdots, \qquad a_i \in \{0, \dots, p-1\}.$$

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Ramification in number theory (continued)

What does $\mathbb{Z}_p[i]$ (or more precisely $\mathbb{Z}_p[i]/(i^2+1)$) look like?

- If $p \equiv 1 \pmod 4$, then it splits as two copies of $\mathbb{Z}_p$. For example, if $p = 5$, then $2 + i$ is divisible by 5 in one copy of $\mathbb{Z}_5$ but is invertible in the other.
- If $p \equiv 3 \pmod 4$, then $\mathbb{Z}_p[i]$ is an integral domain, and every nonzero ideal is a power of (p).
- If $p = 2$, then $\mathbb{Z}_p[i]$ is again an integral domain, but now the ideal $(1 + i)$ is not a power of (2).

We say that 2 is the unique *ramified prime* of the extension $\mathbb{Q}(i)/\mathbb{Q}$ of number fields. Similarly, for any finite extension L/K of number fields, we can identify a finite collection of *ramified prime ideals* of $\mathfrak{o}_K$ (or more precisely, of $\mathfrak{o}_L$).

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Ramification and traces

Let L/K be an extension of number fields, let $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}_K$, and let $\mathfrak{q}$ be a prime ideal of $\mathfrak{o}_L$ appearing in the prime factorization of $\mathfrak{p}$ (i.e, $\mathfrak{q} \cap \mathfrak{o}_K = \mathfrak{p}$). Then $L_\mathfrak{q}$ is a finite extension of $K_\mathfrak{p}$.

For $x \in L_\mathfrak{q}$, the *trace* of x is the trace of multiplication-by-x as a $K_\mathfrak{p}$-linear transformation on $L_\mathfrak{q}$. (It is also the sum of the Galois conjugates of x.)

Lemma

The trace map takes $\mathfrak{o}_{L_\mathfrak{q}}$ into $\mathfrak{o}_{K_\mathfrak{p}}$, and is surjective if and only if $\mathfrak{q}$ is not a ramified prime.

For example, take $K = \mathbb{Q}$, $L = \mathbb{Q}(i)$, $\mathfrak{p} = (2)$, $\mathfrak{q} = (1+i)$. Then

$$\mathrm{Trace}(a+bi) = (a+bi) + (a-bi) = 2a$$

so the image of $\mathrm{Trace} : \mathbb{Z}_2[i] \to \mathbb{Z}_2$ lands in the ideal (2). By contrast, if $\mathfrak{p} = (p)$ for $p \equiv 3 \pmod 4$, then 2 is invertible in $\mathbb{Z}_p$; if $p \equiv 1 \pmod 4$, there are two choices for $\mathfrak{q}$, and in both cases $\mathfrak{o}_{L_\mathfrak{q}} \cong \mathfrak{o}_{K_\mathfrak{p}} \cong \mathbb{Z}_p$.

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Almost disappearing ramification

Tate discovered that ramification "almost disappears" if one makes certain *infinite* extensions of number fields.

E.g., for any prime p, let $\mathbb{Z}_p[\zeta_{p^\infty}]$ (resp. $\mathbb{Q}_p(\zeta_{p^\infty})$) be the ring obtained from $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) by adjoining primitive p^n-th roots of unity ζ_{p^n} for all n. (Note that $\mathbb{Z}_p[\zeta_{p^\infty}]$ is the integral closure of $\mathbb{Z}_p$ in $\mathbb{Q}_p(\zeta_{p^\infty})$.)

Theorem (example of a theorem of Tate)

Let F be a finite extension of $\mathbb{Q}_p(\zeta_{p^\infty})$. Let $\mathfrak{o}_F$ be the integral closure of $\mathbb{Z}_p$ in F. Then $\mathrm{Trace} : \mathfrak{o}_F \to \mathbb{Q}_p(\zeta_{p^\infty})$ *has image in $\mathbb{Z}_p[\zeta_{p^\infty}]$ and is* **almost** *surjective: its image contains the maximal ideal of $\mathbb{Z}_p[\zeta_{p^\infty}]$.*

That maximal ideal is generated by $1 - \zeta_{p^n}$ for $n = 1, 2, \ldots,$ and

$$(1-\zeta_p)^{p-1} = p \times (\text{unit}), \qquad (1-\zeta_{p^{n+1}})^p = (1-\zeta_{p^n}) \times (\text{unit}).$$

So any element "divisible by p^ϵ for some $\epsilon > 0$" is a trace.

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E.g., for any prime p, let $\mathbb{Z}_p[\zeta_{p^\infty}]$ (resp. $\mathbb{Q}_p(\zeta_{p^\infty})$) be the ring obtained from $\mathbb{Z}_p$ (resp. $\mathbb{Q}_p$) by adjoining primitive p^n-th roots of unity ζ_{p^n} for all n. (Note that $\mathbb{Z}_p[\zeta_{p^\infty}]$ is the integral closure of $\mathbb{Z}_p$ in $\mathbb{Q}_p(\zeta_{p^\infty})$.)

Theorem (example of a theorem of Tate)

Let F be a finite extension of $\mathbb{Q}_p(\zeta_{p^\infty})$. Let $\mathfrak{o}_F$ be the integral closure of $\mathbb{Z}_p$ in F. Then $\mathrm{Trace} : \mathfrak{o}_F \to \mathbb{Q}_p(\zeta_{p^\infty})$ has image in $\mathbb{Z}_p[\zeta_{p^\infty}]$ and is **almost** *surjective: its image contains the maximal ideal of $\mathbb{Z}_p[\zeta_{p^\infty}]$.*

That maximal ideal is generated by $1 - \zeta_{p^n}$ for $n = 1, 2, \ldots$, and

$$(1-\zeta_p)^{p-1} = p \times (\mathrm{unit}), \qquad (1-\zeta_{p^{n+1}})^p = (1-\zeta_{p^n}) \times (\mathrm{unit}).$$

So any element "divisible by p^ϵ for some $\epsilon > 0$" is a trace.

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The field of norms equivalence

Fontaine put Tate's result into a broader context, by (partially) isolating a key relevant property of the infinite extensions appearing in Tate's theorem. In the process, he discovered an amazing relationship between Galois theory in characteristic 0 and characteristic p.

Theorem (example of a theorem of Fontaine-Wintenberger)

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Artin-Schreier theory

One consequence of the previous theorem is that one can use special tools from characteristic p to study Galois theory in characteristic 0. One of the most powerful of these is the Artin-Schreier construction.

For any ring R of characteristic p, the map $x \mapsto x^p$ on R is a ring homomorphism, called *Frobenius* and denoted φ.

For any R-module M, we can twist the action of R on M to define a new R-module $M \otimes_\varphi R$:

$$rm \otimes 1 = m \otimes \varphi(r).$$

We define a *φ-module* over R to be a finite projective R-module M equipped with an R-linear isomorphism $F : M \otimes_\varphi R \cong M$. The map $\Phi : M \to M$ given by $\Phi(m) = F(m \otimes 1)$ is *φ-semilinear*:

$$\Phi(r_1 m_1 + r_2 m_2) = \varphi(r_1)\Phi(m_1) + \varphi(r_2)\Phi(m_2).$$

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φ-modules and Galois representations

Let F be a field of characteristic p with absolute Galois group G_F. The group G_F is a compact profinite topological group, and the Galois groups of finite separable extensions of E occur as open subgroups.

Theorem (Katz)

The functors

$$V \mapsto (V \otimes_{\mathbb{F}_p} F^{\text{sep}})^{G_F}, \qquad D \mapsto (D \otimes_F F^{\text{sep}})^{\varphi}$$

define equivalences of categories

$$\left\{ \begin{matrix} \text{continuous}^a \; G_F\text{-representations} \\ \text{on finite } \mathbb{F}_p\text{-vector spaces } V \end{matrix} \right\} \leftrightarrow \{\varphi\text{-modules } D \text{ over } F\}.$$

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Fontaine also introduced the concept of a (φ, Γ)-*module*, which allows the theorem of Katz to be applied to finite extensions of $\mathbb{Q}_p$ (whose absolute Galois groups do not directly appear in characteristic p).

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Almost purity

Faltings discovered that the results of Tate and Fontaine-Wintenberger generalize to a large class of rings which are not fields.

For example, let R be the p-adic completion of

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What is really going on here?

In the mid-2000s, I tried to understand the proofs of the theorems of Fontaine-Wintenberger and Faltings. Failing to do so, I decided to try to come up with my own proofs. Some progress was reported at ICM 2010; subsequent progress includes joint work with Ruochuan Liu.

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A construction of Fontaine

For any ring R, the ring $R/(p)$ is of characteristic p, so it has a Frobenius map φ. We can then define

$$R' = \varprojlim_{\varphi} R/(p);$$

that is, R' consists of sequences $(\ldots, x_1, x_0)$ in $R/(p)$ such that $x_{n+1}^p = x_n$. The ring R' is again of characteristic p, but now φ is bijective: its inverse is the map $(\ldots, x_1, x_0) \to (\ldots, x_2, x_1)$.

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If R is the integral closure of $\mathbb{Z}_p$ in an algebraic closure of $\mathbb{Q}_p$, then R' is a valuation ring in an algebraically closed field.

But what about smaller R? For example, if $R = \mathbb{Z}_p[\zeta_{p^\infty}]$, then R' is the π-adic completion of $\mathbb{F}_p[\pi, \pi^{1/p}, \ldots]$.

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The following is in some sense a *maximal* generalization of the Fontaine-Wintenberger theorem.

Theorem (K-Liu, Scholze)

Let K be the completion of an algebraic extension of $\mathbb{Q}_p$. Suppose that:

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Then the ring $(\mathfrak{o}_K)'$ is a valuation ring in a field K' of characteristic p, and there is a canonical isomorphism of Galois groups $G_K \cong G_{K'}$.

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More on the Fontaine-Wintenberger equivalence

For any ring S of characteristic p on which φ is bijective, there is a unique p-adically separated and complete ring $W(S)$ such that $W(S)/(p) \cong S$. It can be built explicitly using *Witt vectors*. For instance, if $S = \mathbb{F}_p$, then $W(S) \cong \mathbb{Z}_p$, but with a more exotic description.

Fontaine showed that for R, R' as before, there is an explicit map $\theta : W(R') \to R$. The conditions of the theorem imply that $\theta : W(\mathfrak{o}_{K'}) \to \mathfrak{o}_K$ is surjective. One then shows that for every finite extension L' of K',

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The Fontaine construction for rings

Let R be a ring which is p-torsion-free and integrally closed in $R[1/p]$ (e.g., we rule out $\mathbb{Z}_p[px, x^2]$ because x is missing). Suppose also that:

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Again, the map $W(R') \to R$ is surjective. Let $\pi \in R'$ be any element of the form $(\dots, x, p)$; we will then compare $R[1/p]$ and $R'[1/\pi]$ in the same way that we compared K and K' in the previous theorem.

For example, the rings from the example of Faltings's theorem:

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Let R, R' be as on the previous slide.

(a) *If R is p-adically separated and complete*[a]*, then there is an explicit equivalence of categories of finite étale algebras over $R[1/p]$ and $R'[1/\pi]$.*

(b) *Let S be the integral closure of R in some finite étale $R[1/p]$-algebra. Then $\mathrm{Trace} : S \to R$ is almost surjective.*

[a] Or even just henselian with respect to p.

Again, (b) follows from (a) using the surjection $\theta : W(R') \to R'$ and the map φ^{-1} on $R'[1/\pi]$. To prove (a), one reduces to the preceding theorem using a glueing argument in nonarchimedean analytic geometry.

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(b) *Let S be the integral closure of R in some finite étale $R[1/p]$-algebra. Then $\mathrm{Trace} : S \to R$ is almost surjective.*

[a] Or even just henselian with respect to p.

Again, (b) follows from (a) using the surjection $\theta : W(R') \to R'$ and the map φ^{-1} on $R'[1/\pi]$. To prove (a), one reduces to the preceding theorem using a glueing argument in nonarchimedean analytic geometry.

The Faltings almost purity theorem revisited

The following theorem is in some sense a *maximal* generalization of the Faltings almost purity theorem.

Theorem (K-Liu, Scholze)

Let R, R' be as on the previous slide.

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5. 2000s
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Localization and globalization

To *globalize* the previous constructions, one must first *localize* them. This fails in the category of schemes: for R, R' as before, there is no canonical homeomorphism of the Zariski spectra (prime ideals) of R and R'. But:

Theorem (K)

There is a canonical homeomorphism of the Berkovich-Gelfand spectra (real valuations) of R and R'.

Theorem (K-Liu, Scholze)

There is a canonical homeomorphism of the Huber adic spectra (arbitrary-rank valuations) of R and R'.

One thus has a single topological space equipped with two sheaves of rings which are of different characteristics, but are closely related! Example: by almost purity, both sheaves yield the same *étale topology*.

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Perfectoid spaces and applications

The resulting spaces have numerous uses:

- New cases of the weight-monodromy conjecture (Scholze).
- Parametrization of p-divisible groups (Scholze-Weinstein).
- Stable reduction of modular curves (Weinstein, Scholze).
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