An overview of the p-adic local Langlands correspondence (after Colmez)

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The global Langlands correspondence

Let F be a global field (a number field or a finite extension of $\mathbb{F}_p(t)$ for some prime p). Fix a prime number ℓ which is nonzero in F.

The global Langlands correspondence for the group GL_n is supposed to relate continuous representations of the absolute Galois group G_F on n-dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces (unramified away from finitely many places of F) with automorphic representations of the adelic group $\mathrm{GL}_n(\mathbb{A}_F)$. Under this correspondence, the spectrum of Frobenius at a place v of F on the Galois side is supposed to match the spectrum of the Hecke operator at v on the automorphic side.

The case $n = 1$ reproduces class field theory.

The local Langlands correspondence ($\ell \neq p$)

Let F be a local field (a complete discretely valued field with finite residue field). Fix a prime number ℓ which is nonzero in the residue field of F.

The local Langlands correspondence for the group GL_n relates continuous representations of G_F on n-dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces (with no condition on ramification) with representations of the adelic group $\mathrm{GL}_n(\mathbb{A}_F)$.

There is supposed to be **local-global compatibility** of the Langlands correspondence: for F a global field and v a place of F, restricting from G_F to G_{F_v} on the Galois side is supposed to correspond to restricting from $\mathbb{A}_F$ to its factor F_v.

Again, the case $n = 1$ reproduces class field theory.

The local Langlands correspondence ($\ell = p$)

Let F be a local field of residue characteristic p.

One can ask whether there is a meaningful version of the local Langlands correspondence in this setting which exhibits local-global compatibility. Because there are "many" continuous representations of G_F on finite-dimensional $\overline{\mathbb{Q}}_p$-vector spaces, one must rigidify this question by asking for compatibility with p-adic analytic interpolation.

One would also like some sort of compatibility with reduction modulo p.

The case of $\mathrm{GL}_2(\mathbb{Q}_p)$

A miracle happens for the group GL_2 over $\mathbb{Q}_p$: Colmez has constructed a correspondence that does everything one would want. (The existence of same had been conjectured by Breuil.)

This uses Fontaine's theory of (φ, Γ)-**modules**, which gives a convenient alternate description of the category of continuous representations of $G_{\mathbb{Q}_p}$ on finite-dimensional $\mathbb{Q}_p$-vector spaces. More on this description shortly.

Reading off the Langlands correspondence from a (φ, Γ)-module is in the same spirit as other constructions in p-adic Hodge theory; for example, given the p-adic étale cohomology of $X_{\overline{\mathbb{Q}_p}}$ for some smooth proper scheme X over $\mathbb{Z}_p$, one can read off the comparison isomorphism with crystalline cohomology (Berger).

What is a (φ, Γ)-module? (version 1)

Let $\mathbf{A}$ be the p-adic completion of $\mathbb{Z}_p((\pi))$. This ring admits an endomorphism φ, and automorphisms indexed by $\gamma \in \Gamma = \mathbb{Z}_p^\times$, characterized by

$$\varphi(1+\pi) = (1+\pi)^p,$$
$$\gamma(1+\pi) = (1+\pi)^\gamma = \sum_{n=0}^{\infty} \frac{\gamma(\gamma-1)\cdots(\gamma-n+1)}{n!}\pi^n$$

plus continuity for the inverse limit topology given by putting the π-adic topology on $\mathbb{Z}/p^n\mathbb{Z}((\pi))$.

A **(projective, étale)** (φ, Γ)**-module** over $\mathbf{A}$ is a finite free $\mathbf{A}$-module M equipped with commuting semilinear continuous actions of φ and Γ, for which the induced map $\varphi^* M \to M$ is an isomorphism.

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Theorem (Fontaine)

There is an explicit equivalence of categories between the category of (φ, Γ)*-modules over* $\mathbf{A}$ *and the category of continuous representations of* $G_{\mathbb{Q}_p}$ *on finite free* $\mathbb{Z}_p$*-modules.*

What is a (φ, Γ)-module? (version 2)

Let $\mathbf{A}^\dagger$ be the subring of $\mathbf{A}$ consisting of series which converge in some region of the form $* < |\pi| < 1$ (such series are said to be **overconvergent**); this subring is stable under φ and Γ. Define a (φ, Γ)-**module** over $\mathbf{A}^\dagger$ using the same recipe as over $\mathbf{A}$.

Theorem (Cherbonnier–Colmez)

The categories of (φ, Γ)-modules over $\mathbf{A}^\dagger$ and $\mathbf{A}$ are equivalent via base extension. In particular, by Fontaine they are both equivalent to the category of continuous representations of $G_{\mathbb{Q}_p}$ on finite free $\mathbb{Z}_p$-modules.

A monoid algebra reinterpreted (part 2)

It is natural to view φ and Γ together as forming a single commutative monoid isomorphism to $\mathbb{Z}_p \setminus \{0\}$, acting on $\mathbf{A}$ and $\mathbf{A}^\dagger$ as

$$x(1+\pi) = (1+\pi)^x.$$

Consequently, any (φ, Γ)-module may be viewed as a left module for the twisted monoid algebra $\mathbf{A}\langle \mathbb{Z}_p \setminus \{0\}\rangle$.

Thanks to the continuity condition, we can extend the action of $\mathbf{A}\langle \mathbb{Z}_p \setminus \{0\}\rangle$ to a larger ring.* Namely, identify the Iwasawa algebra $\mathbb{Z}_p[\![1+p\mathbb{Z}_p]\!]$ with $\mathbb{Z}_p[\![T]\!]$. We then define a ring structure on $\mathbb{Z}_p[\![\pi, T]\!]$ so

$$(1+T)^{-1}(1+\pi)(1+T) = (1+\pi)^{1+p},$$

invert π, impose the π-adic topology modulo each power of p, and take the inverse limit.

*For convenience, assume $p > 2$.

A monoid algebra reinterpreted (part 2)

The same ring can be desired from the untwisted monoid algebra over $\mathbb{Z}_p$ for the monoid

$$\begin{pmatrix} \mathbb{Z}_p \setminus \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$$

by identifying $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ with $(1+\pi)^x$.

Automorphic representations from (φ, Γ)-modules

As noted above, from a (φ, Γ)-module M we obtain an action of the monoid

$$\begin{pmatrix} \mathbb{Z}_p \setminus \{0\} & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}.$$

Let $\psi : M \to M$ be the reduced trace† of φ. By replacing M with $\varprojlim_\psi M$, we obtain an object with an action of the group

$$\begin{pmatrix} \mathbb{Q}_p^\times & \mathbb{Q}_p \\ 0 & 1 \end{pmatrix}.$$

By a suitable induction from this subgroup (the *mirabolic*) to $\mathrm{GL}_2(\mathbb{Q}_p)$, we obtain Colmez's candidate for the Galois-to-automorphic construction. (One checks that this works by looking carefully at certain specializations.)

†This map *a priori* is valued in $M[1/p]$, but it does in fact land in M.

Overconvergent descent and locally analytic vectors

As noted earlier, a (φ, Γ)-module over $\mathbf{A}$ descends canonically to the subring $\mathbf{A}^\dagger$. By tracing this descent through Colmez's construction, we obtain a subrepresentation of $\mathrm{GL}_2(\mathbb{Q}_p)$.

This turns out to be the **locally analytic** vectors in the original representation. The key point is that $\mathbf{A}^\dagger$, while dense in $\mathbf{A}$ for the inverse limit topology, admits an alternate topology under which $\mathbf{A}^\dagger$ is complete **and** the action of Γ remains continuous.

Okay, now what?

So far so good, but how do we go past $\mathrm{GL}_2(\mathbb{Q}_p)$?

If F is a finite extension of $\mathbb{Q}_p$, we can exhibit a similar theory of (φ, Γ)-modules associated to representations of G_F. However, this fails to reproduce the previous success for two reasons.

- The base ring will be a finite étale algebra over the previous one. In particular, it will still correspond to completing a group algebra for a commutative p-adic Lie group of rank 1 over $\mathbb{Q}_p$ (not F).
- The group Γ will be replaced by a subgroup of finite index. In particular, it will remain a p-adic Lie group of rank 1 over $\mathbb{Q}_p$ (not F).

Similar issues arise if we try to replace GL_2 with a group of higher rank, such as GL_n.

Beyond $\mathrm{GL}_2(\mathbb{Q}_p)$

In order to go further, we need additional constructions of **multivariate** (φ, Γ)-modules associated to Galois representations.

In order to follow recent developments in the Langlands correspondence (especially the work of V. Lafforgue), one must also find ways to characterize representations of **products** of Galois groups.

Ingredients

The action of Γ in the usual theory of (φ, Γ)-modules is derived from the action of $\mathbb{Z}_p^\times$ on $\mathbb{Q}_p(\mu_{p^\infty})$. One can construct a parallel theory for any infinitely ramified p-adic Lie extension of any finite extension of $\mathbb{Q}_p$ **except** that this happens in the language of perfectoid rings, which does not provide access to locally analytic vectors.

To do that, one must establish variants of the Cherbonnier-Colmez overconvergent descent. Recent progress has been made on this by H. Gao and T. Liu.

It seems easier to extend overconvergent descent to products; this takes advantage of a construction of Drinfeld. (Joint work with Carter and Zábrádi.)

For more information...

- KSK, Frobenius modules over multivariate Robba rings (arXiv:1311.7468v2).
- A. Carter, KSK, and G. Zábrádi, Drinfeld's lemma for perfectoid spaces and overconvergence of multivariate (φ, Γ)-modules (arXiv:1808.03964v2).

To be continued next summer in Budapest!