Companions in étale cohomology

Kiran S. Kedlaya

Department of Mathematics, University of California, San Diego
School of Mathematics, Institute for Advanced Study (2018–2019)
kedlaya@ucsd.edu
http://kskedlaya.org/slides/

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Contents

Zeta functions of algebraic varieties

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$$Z(X, T) = \prod_{x \in X^\circ} (1 - T^{\deg(x/\mathbb{F}_q)})^{-1} = \exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} \# X(\mathbb{F}_{q^n})\right),$$

where X° denotes the closed points of X (i.e., Galois orbits of $\overline{\mathbb{F}}_q$-points).

- For $X = \mathbb{A}^n_{\mathbb{F}_q}$ (affine space), $Z(X, T) = \frac{1}{1 - q^n T}$.
- For $X = V \sqcup W$ (disjoint union), $Z(X, T) = Z(V, T) Z(W, T)$.
- For $X = \mathbb{P}^n_{\mathbb{F}_q}$ (projective space), $Z(X, T) = \frac{1}{(1-T)(1-qT)\cdots(1-q^n T)}$.
- For X an elliptic curve, $Z(X, T) = \frac{1 - aT + qT^2}{(1-T)(1-qT)}$ with $a \in \mathbb{Z} \cap [-2\sqrt{q}, 2\sqrt{q}]$ (Hasse).

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Smoothness and irreduciblity

In order to be more precise about the properties of $Z(X, T)$, we impose some additional restrictions on X hereafter.

- X must be *smooth*: it can be described as an intersection of hypersurfaces in an affine space whose gradients are linearly independent.
- X must be *geometrically irreducible*: it cannot be written as the union of two closed subvarieties, and likewise after enlarging $\mathbb{F}_q$.

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The Weil conjectures

Based on key examples (e.g., Fermat hypersurfaces) and an analogy with Riemann/Dedekind zeta functions in number theory, Weil conjectured:

- $Z(X, T)$ is always a rational function;
- if X is projective[†] over $\mathbb{F}_q$, $Z(X, T)$ factors as

$$\frac{P_1(T) \dots P_{2\dim(X)-1}(T)}{P_0(T) \cdots P_{2\dim(X)}(T)}$$

  where $P_i(T) \in 1 + T\mathbb{Z}[T]$ has roots in $\mathbb{C}$ on the circle $|T| = q^{-i/2}$;
- if in addition X admits a smooth lift to characteristic 0, then $\deg(P_i)$ equals the i-th Betti number of any lift.

These results were first proved using *étale cohomology*, in a series of works culminating in the work of Deligne (mid-1970s).

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The fundamental group of an algebraic variety

Let $\mathbb{F}_q(X)$ denote the field of rational functions on X. Let $G_{\mathbb{F}_q(X)}$ be the Galois group of $\mathbb{F}_q(X)$ (i.e., the group of automorphisms of a separable algebraic closure of $\mathbb{F}_q(X)$).

For each $x \in X^\circ$, let $\kappa(x)$ be the residue field of X at x, and let $K(x)$ be the completion of $\mathbb{F}_q(X)$ at x (i.e., take the local ring of X at x, complete for the maximal ideal, then take the fraction field). Then $G_{K(x)}$ injects into $G_{\mathbb{F}_q(X)}$ and surjects onto $G_{\kappa(x)}$; the kernel of $G_{K(x)} \to G_{\kappa(x)}$ is called I_x (the *inertia group* of x).

The *étale fundamental group* $\pi_1(X)$ is the quotient of $G_{\mathbb{F}_q(X)}$ by the smallest closed normal subgroup containing I_x for each $x \in X^\circ$. Like $G_{\mathbb{F}_q(X)}$, it is a profinite topological group.

This still makes sense if we replace $\mathbb{F}_q$ with $\mathbb{C}$. In this case, we get the profinite completion of the topological fundamental group.

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Local systems in algebraic geometry

Let ℓ be a prime *other than p*. By an *ℓ-adic local system* on X, I will mean a continuous representation $\mathcal{E}$ of $\pi_1(X)$ on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space.

For any morphism $Y \to X$ of varieties, I get a morphism $\pi_1(Y) \to \pi_1(X)$, so I can pull back an ℓ-adic local system on X to an ℓ-adic local system on Y. In particular, if $Y = \{x\}$ is a closed point, then I get a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space with the action of $G_{\kappa(x)}$, which has a distinguished generator (Frobenius). Let $P(\mathcal{E}, x)$ be the characteristic polynomial of the action of this generator.

One can define *étale cohomology* with coefficients in a local system by taking continuous group cohomology. This construction powers most of the following discussion, but we won't see much of it explicitly.

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A conjecture of Deligne: motivation

An ℓ-adic local system $\mathcal{E}$ on X *comes from geometry* if it appears in the relative étale cohomology of some smooth proper morphism $f : Y \to X$. Concretely, this means that for each $x \in X^\circ$, $P(\mathcal{E}, x)$ shows up as a factor in the zeta function of the fiber of f over x.

As a consequence of the Weil conjectures, ℓ-adic local systems that come from geometry have very strong arithmeticity properties. Deligne has conjectured that every ℓ-adic local system on X looks like it "comes from geometry", up to separating into irreducibles and taking twists.

To rigidify the situation enough to formulate the conjecture, we assume that $\mathcal{E}$ is irreducible and that its determinant is a character of finite order; this second restriction eliminates most twists.

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Conjecture (Deligne, 1978)

Let $\mathcal{E}$ be an ℓ-adic local system on X. Assume that $\mathcal{E}$ is irreducible and its determinant is of finite order.

(i) *The roots of $P(\mathcal{E}, x)$ in $\overline{\mathbb{Q}}_\ell$ are all algebraic over $\mathbb{Q}$, and their conjugates in $\mathbb{C}$ all lie on the unit circle.*

(ii) *The coefficients of $P(\mathcal{E}, x)$ lie in a number field depending only on $\mathcal{E}$.*

(iii) *The roots of $P(\mathcal{E}, x)$ are p-units (i.e., integral over $\mathbb{Z}[p^{-1}]$).*

(iv) *For any p-adic valuation v on $\overline{\mathbb{Q}}$, the roots of $P(\mathcal{E}, x)$ all have valuation in the range $[-\frac{1}{2}\operatorname{rank}(\mathcal{E})v(\#\kappa(x)), \frac{1}{2}\operatorname{rank}(\mathcal{E})v(\#\kappa(x))]$.*

(v) *Fix a prime $\ell' \neq p$ and an isomorphism of $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_\ell$ with $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_{\ell'}$. Then there exists an ℓ'-adic local system $\mathcal{E}'$ on X such that $P(\mathcal{E}, x) = P(\mathcal{E}', x)$ for all $x \in X^\circ$. (We say $\mathcal{E}'$ is a* **companion** *of $\mathcal{E}$.)*

(vi) *The analogue of (v) holds for $\ell' = p$, whatever that means. (Deligne: "On espére des petits camarades cristallins.")*

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p-adic analysis and zeta functions

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This and subsequent developments inspired Grothendieck to propose *crystalline cohomology* as a p-adic cohomology theory which could fill in for étale cohomology when $\ell = p$. It is modeled on cohomology of differential forms (de Rham cohomology).

Crystalline cohomology only behaves well for (smooth) projective[‡] varieties. The right cohomology to use for general smooth varieties is *rigid cohomology*, defined by Berthelot based on a Dwork-style construction of Monsky–Washnitzer for affine varieties (*formal cohomology*).

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The p-adic analogue of a local system

Berthelot's theory includes a p-adic analogue of an ℓ-adic local system, called an *overconvergent F-isocrystal*. Such an object is (very loosely) a vector bundle equipped with an integrable connection.

Crew was the first to suggest that overconvergent F-isocrystals could act as the missing objects (the "petits camarades cristallins") in part (vi) of Deligne's conjecture. One can also formulate Deligne's conjecture with $\ell = p$, taking $\mathcal{E}$ to be an overconvergent F-isocrystal.

Since the mid-1980s, it has been expected that rigid cohomology obeys formal properties analogous to those of étale cohomology, to the extent that (for example) it could be used to rederive the Weil conjectures. These formal properties are now known, thanks to difficult work by many authors (Abe, Berthelot, Chiarellotto, Caro, Crew, de Jong, K, Shiho, Tsuzuki).

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What is... an overconvergent F-isocrystal?

For X (smooth) affine, one can define[§] overconvergent F-isocrystals and their cohomology in terms of a smooth lift $\mathfrak{X}$ of X over some finite extension of $\mathbb{Z}_p$.

For concreteness, take the example $X = \mathbb{A}^n_{\mathbb{F}_q}$. Let K be a finite extension of $\mathbb{Q}_p$ with residue field containing $\mathbb{F}_q$ and take $\mathfrak{X} = \mathbb{A}^n_{\mathfrak{o}_K}$. The Raynaud generic fiber of $\mathfrak{X}$ is the closed unit n-disc over K; take a vector bundle $\mathcal{E}$ on it equipped with an integrable connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_{X/K}$. This approximately[¶] defines a *convergent*[‖] *isocrystal* on X.

To add F, pick a morphism $\sigma : \mathfrak{X} \to \mathfrak{X}$ lifting the absolute Frobenius map $x \mapsto x^p$ on X (e.g., in this example, take each coordinate to its p-th power). Then a *convergent F-isocrystal* is a convergent isocrystal $\mathcal{E}$ together with an isomorphism $F : \sigma^*\mathcal{E} \to \mathcal{E}$ that respects connections.

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There is an exact sequence

$$1 \to \pi_1(X_{\overline{\mathbb{F}}_q}) \to \pi_1(X) \to G_{\mathbb{F}_q} \to 1.$$

The group $\pi_1(X_{\overline{\mathbb{F}}_q})$ is often easier to describe (e.g., for a projective curve of genus g it is profinite on $2g$ generators with one relation) and has many representations which cannot be extended to $\pi_1(X)$; the interaction with $G_{\mathbb{F}_q}$ imposes severe constraints.

Analogously, there exist lots of (over)convergent isocrystals, because they can be described using only one structure (an integrable connection); however, (over)convergent F-isocrystals are specified using two separate structures which must be compatible in a nontrivial way.

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where $\nabla^{(2)}(f \otimes dg) = \nabla(f) \wedge dg$. The cohomology** of this complex is the *rigid cohomology of X with coefficients in $\mathcal{E}$* (or for short, the cohomology of $\mathcal{E}$). These are finite-dimensional K-vector spaces (K, 2006).

To compare with ℓ-adic cohomology, we want a K-linear action of F, but right now this is off by an automorphism of K. We fix this by tensoring over $\mathbb{Q}_p$ with $\overline{\mathbb{Q}}_p$ and taking fixed subspaces for the automorphism of K (acting trivially on $\overline{\mathbb{Q}}_p$); this gives an $\overline{\mathbb{Q}}_p$-vector space with linear F-action.

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The Langlands correspondence for function fields

Let X be a curve over $\mathbb{F}_q$, with smooth compactification $\overline{X}$. Then $\mathbb{F}_q(X)$ is a function field of transcendence degree 1, and as such is strongly analogous to a number field. In particular, we may define its adèle ring $\mathbb{A}_{\mathbb{F}_q(X)}$ by taking a restricted product of its completions at all places.

For any $\ell \neq p$, Langlands conjectured[††] a correspondence between rank-n irreducible ℓ-adic local systems with determinant of finite order and cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{F}_q(X)})$ unramified on X, in which $P(\mathcal{E}, x)$ matches the characteristic polynomial of a Hecke operator attached to x.

For $n = 1$, this reproduces class field theory for $\mathbb{F}_q(X)$. For $n = 2$, this was shown by Drinfeld; for $n > 2$, this was done by L. Lafforgue.

One also has a similar correspondence involving overconvergent F-isocrystals. This was shown by T. Abe, emulating Lafforgue.

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The Langlands correspondence and geometric origins

As stated, the Langlands correspondence is not enough information for our purposes. However, the proof gives much more: it shows that ℓ-adic local systems and overconvergent F-isocrystals come from geometry, specifically from moduli spaces of *shtukas* (certain vector bundles with extra structure); these act like Shimura varieties.

Combining this with the Langlands bijection, the matching of characteristic polynomials (plus a similar statement about points of $\overline{X} \setminus X$), and properties of étale and rigid cohomology, one gets a complete resolution of Deligne's (updated) conjecture when X is a curve.

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For X of dimension > 1, there is not even a conjectural analogue of the Langlands correspondence that would provide a uniform source of geometric origins for local systems. Even the analogue of class field theory is quite subtle, and is a topic of active research.

Instead, we use the case of curves as a black box. For every curve in X, we can use the correspondence to gather information; in addition, when two curves cross at x, the values of $P(\mathcal{E}, x)$ on the two curves coincide.

For the components of Deligne's conjecture that concern valuations of roots of $P(\mathcal{E}, x)$ at individual points (parts (i), (iii), (iv)), this is essentially enough; one only[‡‡] needs to check that $\mathcal{E}$ "usually" remains irreducible after restricting to a curve.

For the remaining components, more work is required; the general theme is to exploit *uniformity* statements quantified over curves in X.

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Deligne's finiteness argument

Part (ii) of Deligne's conjecture states that the $P(\mathcal{E}, x)$ all have coefficients in a single number field. Deligne (2012) shows this by showing that $\mathcal{E}$ is uniquely determined (up to semisimplification) by its $P(\mathcal{E}, x)$ for x of degree up to some explicit bound. (By Galois theory, the field generated by the coefficients of those $P(\mathcal{E}, x)$ contains all the rest.)

This can be done by restricting to curves, making sure that the cutoff value can be chosen uniformly. This case is treated by a direct calculation involving L-functions. A key point is that by replacing X with some finite étale cover, one can kill all wild ramification of $\mathcal{E}$, and thus control the Euler characteristic by the Grothendieck–Ogg–Shafarevich formula.

This argument adapts easily to the case $\ell = p$, modulo the key point: for $\ell \neq p$, this is handled by choosing a lattice in the associated representation and trivializing it mod ℓ; any remaining ramification is of ℓ-power order and hence tame. For $\ell = p$, something similar is true but deep (K, 2011).

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Part (v) of Deligne's conjecture involves constructing an ℓ'-adic companion of $\mathcal{E}$. Since such an object is a representation of $\pi_1(X)$, we can try to do it by building a compatible family of mod-$(\ell')^n$ representations.

Drinfeld (2012) proves this by interpolating this family from the restrictions to curves, where we have such families using the Langlands correspondence. The key point is that modulo any fixed power of ℓ', one can build a *finite* family of representations such that for any curve, one of these has the right restriction. Then an easy compactness argument lets you put together the compatible family.

It is relatively easy to adapt this to the case where $\mathcal{E}$ is an overconvergent F-isocrystal, since we are still assuming $\ell' \neq p$. This was carried out (independently) by Abe–Esnault and K (2017).

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However, one can still execute a form of Drinfeld's strategy (K, 2019). One again takes an "exhaustive" family of curves (built using a variant of Poonen's Bertini theorem over finite fields), form the companion on each of those using the Langlands correspondence, pick integral structures, reduce modulo powers of p, and establish a finiteness property (this time using slopes of vector bundles).

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(vi) *The analogue of (v) holds for $\ell' = p$, taking $\mathcal{E}'$ to be an overconvergent F-isocrystal.*

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Theorem (L. Lafforgue, Abe, Deligne, Drinfeld, Abe–Esnault, K)

Let $\mathcal{E}$ be an ℓ-adic local system or an overconvergent F-isocrystal on X. Assume that $\mathcal{E}$ is irreducible and its determinant is of finite order.

(i) *The roots of $P(\mathcal{E}, x)$ in $\overline{\mathbb{Q}}_\ell$ are all algebraic over $\mathbb{Q}$, and their conjugates in $\mathbb{C}$ all lie on the unit circle.*

(ii) *The coefficients of $P(\mathcal{E}, x)$ lie in a number field depending only on $\mathcal{E}$.*

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Contents

Valuations of Frobenius eigenvalues

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By contrast, one can only control ℓ-adic valuations in étale cohomology and p-adic valuations in rigid cohomology. In particular, part (v) of the conjecture implies part (iii), while part (vi) implies part (iv) (here using a result of Drinfeld-K).

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Newton polygons

Let v be a p-adic valuation on $\overline{\mathbb{Q}}_p$. For each $x \in X^\circ$, let $N(\mathcal{E}, x, v)$ denote the *(normalized) Newton polygon* of $P(\mathcal{E}, x, v)$; that is, it is the graph of a convex, piecewise linear function from $[0, \mathrm{rank}(\mathcal{E})]$ to $\mathbb{R}$ which has one length-1 interval of slope $v(\alpha)/v(\#\kappa(x))$ for each root α of $P(\mathcal{E}, x)$. (Part (iv) of the conjecture asserts that these slopes are at most $\frac{1}{2}\mathrm{rank}(\mathcal{E})$.)

For $\mathcal{E}$ an overconvergent F-isocrystal, the function $x \mapsto N(\mathcal{E}, x, v)$ is upper semicontinuous (Grothendieck–Katz); in particular, it takes only finitely many values, and each value occurs on a locally closed stratum. Moreover, jumps only occur in codimension 1 (de Jong–Oort).

For $\mathcal{E}$ an ℓ-adic local system, the only way we know how to prove an analogous statement is to verify (vi).

Concrete example: if E is an elliptic curve over $\mathbb{F}_q$ with $q = p^n$ and $a_q = q + 1 - \#E(\mathbb{F}_q)$, then a_q is divisible by p iff it is divisible by $p^{\lfloor n/2 \rfloor}$.

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p-adic companions and L-functions

To any ℓ-adic local system $\mathcal{E}$, we may define its associated L-function

$$L(\mathcal{E}, T) = \prod_{x \in X^\circ} \det(1 - FT, \mathcal{E}_x)^{-1};$$

it is a rational function of T. Note that $\det(1 - FT, \mathcal{E}_x)$ is the reverse of the polynomial $P(\mathcal{E}, x)$; consequently, Deligne's conjecture implies statements about the zeroes/poles of $L(\mathcal{E}, T)$ analogous to the Weil conjectures.

If $\mathcal{E}$ admits a p-adic companion, we can also analyze the *unit-root L-function*, in which (for some p-adic valuation) we retain only the factor of $\det(1 - FT)$ consisting of roots of valuation 0; this is not typically a rational function, but is a ratio of two p-adically entire power series (Dwork's conjecture, proved by Wan).

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Geometric origins

In some cases, the existence of a p-adic companion can be used to establish that an ℓ-adic local system comes from geometry. Specifically, one can show that certain rank-2 local systems arise from families of abelian varieties (Krishnamoorthy-Pál).