Banach bundles

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These slides are available from https://kskedlaya.org/slides/.

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Sheaves in algebraic geometry

For any ring A, the category of A-modules is equivalent to the category of **quasicoherent sheaves** on the topological space $\operatorname{Spec} A$.

A special case of this is that the category of **finite projective** A-modules is equivalent to the category of **locally finite free quasicoherent sheaves** on $\operatorname{Spec} A$. These also correspond to **vector bundles** over $\operatorname{Spec} A$, and I will frequently confuse the two.

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Classical p-adic geometry

Classical p-adic analytic geometry is analogous to the geometry of schemes **locally of finite type** over a field k. These schemes are locally of the form $\operatorname{Spec} A$ where A is a quotient of a polynomial ring $k[x_1, \dots, x_n]$.

Classical p-adic analytic geometry involves spaces locally associated to a quotient of a ring of the form $K\langle x_1, \dots, x_n \rangle$ where K is a field complete with respect to a nonarchimedean absolute value (e.g., $\mathbb{Q}_p$). Here $K\langle x_1, \dots, x_n \rangle$ denotes the subring of $K[\![x_1, \dots, x_n]\!]$ consisting of series which converge for $|x_1|, \dots, |x_n| \leq 1$; it is the completion of $K[x_1, \dots, x_n]$ for the Gauss norm.

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Models of nonarchimedean geometry

There are a variety of recipes to turn such rings into spaces (which all give essentially the same theory). For example, say $A = \mathbb{Q}_p\langle x_1, \dots, x_n \rangle$.

- Tate: maximal ideals of A. This is a subset of $\operatorname{Spec} A$, but with a different Grothendieck topology.
- Raynaud: the formal scheme $\operatorname{Spf} \mathbb{Z}_p[x_1, \dots, x_n]^{\wedge}_{(p)}$ with its topology **modulo** inverting blowups in the special fiber.
- Berkovich: multiplicative seminorms on A. This set (the Gelfand spectrum) has a "reasonable" ordinary topology, but in general we need a finer Grothendieck topology.
- Huber: continuous valuations on A (not necessary real-valued). This set carries a natural **ordinary** topology; it is even a **spectral space**.

Of these, only Huber's approach extends to more general topological rings (Huber's f-adic rings). However, one can do "Huber + Raynaud" (Fujiwara–Kato) or "Huber + Berkovich" (KSK).

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Theorem (Tate, Kiehl)

For A an affinoid algebra over K corresponding to the analytic space X, the category of finite A-modules is equivalent to the category of coherent sheaves on X. Moreover, for any such sheaf $\mathcal{F}$, $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Can this be extended to more general (topological) A-modules and sheaves on X?

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The analytic FF curve

Let F be a perfect field of characteristic $p > 0$ complete with respect to a nontrivial absolute value (i.e., a **perfectoid field** of characteristic p).

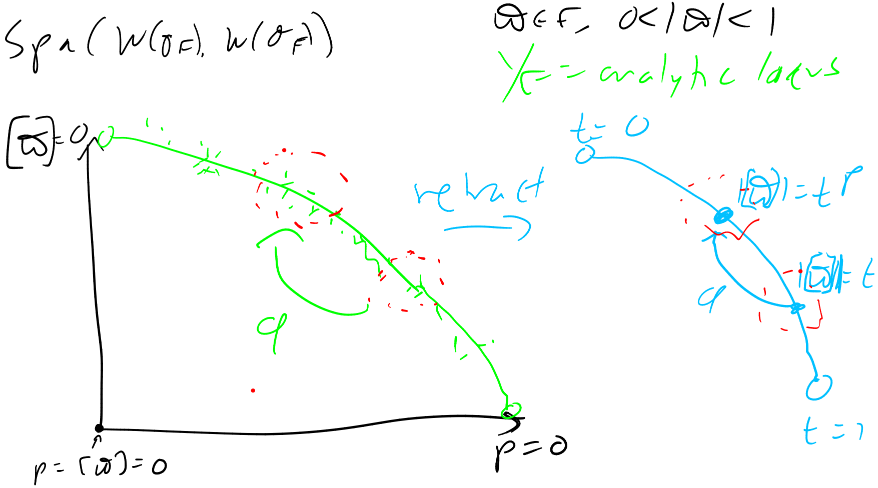
The space Y_F is the “analytification” of $W(\mathfrak{o}_F)$. In Huber's language, it is the subspace of $\mathrm{Spa}(W(\mathfrak{o}_F), W(\mathfrak{o}_F))$ consisting of valuations v with $0 < v(x) < 1$ for all x in the maximal ideal of $W(\mathfrak{o}_F)$.

The Frobenius map $\varphi : W(\mathfrak{o}_F) \to W(\mathfrak{o}_F)$ induces a properly discontinuous[†] automorphism of Y_F. The **analytic Fargues–Fontaine curve** is the quotient $X_F^{\mathrm{an}} = Y_F/\varphi$.

This space is (quasi)compact and noetherian (!).

[†] Reminder: this means that any point has a neighborhood whose translates by distinct powers of φ are pairwise disjoint.

Illustration



The schematic FF curve

We may write down vector bundles on X_F^{an} by writing down vector bundles on Y_F equipped with an action of φ. In particular, taking the trivial bundle with the action of φ on a free generator being multiplication by p^{-n} gives a line bundle $\mathcal{O}(n)$.

The **schematic FF curve** is the scheme

$$X_F = \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} \Gamma(X_F^{\mathrm{an}}, \mathcal{O}(n))\right).$$

This scheme is noetherian (!) and regular of dimension 1 (!!). However, it is **not** of finite type over $\mathbb{Q}_p$, or indeed over any field at all!

By the universal property of affine schemes, there is a natural morphism $X_F^{\mathrm{an}} \to X_F$ in the category of locally ringed spaces.

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Theorem (GAGA, Serre, Grothendieck)

Let X be a proper scheme of finite type over $\mathbb{C}$. Then the morphism $X^{\mathrm{an}} \to X$ of locally ringed spaces induces an equivalence of categories for coherent sheaves. Moreover, sheaf cohomology groups are preserved.

Theorem (Nonarchimedean GAGA)

Let X be a proper scheme of finite type over a nonarchimedean field K. Then the morphism $X^{\mathrm{an}} \to X$ of locally ringed spaces induces an equivalence of categories for coherent sheaves. Moreover, sheaf cohomology groups are preserved.

Theorem (KSK–Liu)

Pullback along the morphism $X_F^{\mathrm{an}} \to X_F$ of locally ringed spaces induces an equivalence of categories for coherent sheaves. Moreover, sheaf cohomology groups are preserved.

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The key lemma for GAGA

Lemma (KSK–Ruochuan Liu)

For any vector bundle (or coherent sheaf) $\mathcal{F}$ on X_F^{an}, for all $n \gg 0$,

(a) $H^1(X_F^{\mathrm{an}}, \mathcal{F}(n)) = 0$;

(b) $\mathcal{F}(n)$ *is generated by global sections.*

In other words, $\mathcal{O}(1)$ is an **ample** line bundle on X_F^{an}.

The proof of (a) amounts to a careful use of the Banach contraction mapping theorem. From (a) it is relatively straightforward to deduce (b).

Warning: $H^0(X_F^{\mathrm{an}}, \mathcal{F}(n))$ is not a finite-dimensional $\mathbb{Q}_p$-vector space! It does have a weaker finiteness property, of being a **Banach–Colmez space**.

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Why this matters

There is a classification[‡] of vector bundles on X_F akin to that for $\mathbb{P}^1$ (K, Hartl–Pink, Fargues–Fontaine, Fargues–Scholze). The semistable bundles of degree 0 are related to p-adic Galois representations (as in Narasimhan–Seshadri over $\mathbb{C}$).

Using the GAGA construction, one can construct **moduli stacks of vector bundles** on X_F (analogous to the use of GIT quotients to construct moduli spaces of vector bundles on curves). These can then be used to construct **moduli spaces of local shtukas** for use in the local Langlands correspondence (Fargues–Scholze).

[‡] For F algebraically closed, not just perfectoid.

The goal of this talk

In classical p-adic (or complex geometry), **analytification** is a functor from schemes locally of finite type over a nonarchimedean field K to analytic spaces over K. If X is such a scheme, then for any analytic space Y, any morphism $Y \to X$ of locally ringed spaces factors through X^{an}.

This functor does **not** apply to X_F because the latter is not of finite type over any field. However, we will show that the GAGA statement for X_F is not isolated! Putting it in the right context is a question for future work.

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Relative versions

GAGA for p-adic analytic spaces can also be formulated for a scheme over an affinoid algebra (Kopf, Conrad).

GAGA for FF curves can also be formulated for the **relative FF curve** over a **perfectoid space** (KSK–Liu). However, the latter is **not** locally noetherian, so we only do the vector bundle case. (This is relevant to studying relative p-adic Galois representations, a/k/a $\mathbb{Q}_p$-**local systems**.)

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Kazuhiro Fujiwara and Fumihiro Kato, *Foundations of Rigid Geometry, I*.

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Banach modules over a Huber ring

Let A be a complete Tate Huber ring. One way to get one of these is to take a (commutative) **Banach ring**, i.e., a ring complete for a submultiplicative norm **and** containing a topologically nilpotent unit, and retain the underlying topology. (E.g., an affinoid algebra.)

A **Banach module** is an A-module which admits a topology with respect to which it is complete and metrizable. (By the Banach open mapping theorem, this topology is unique if it exists.)

Let $\underline{\mathrm{BMod}}_A$ be the category of Banach modules with continuous homomorphisms.

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Projective objects in the category of Banach modules

As usual, a **projective** object P of $\underline{\mathrm{BMod}}_A$ is characterized by this diagram:

$$\begin{array}{ccccc} M & \longrightarrow & N & \longrightarrow & 0 \\ \uparrow & \nearrow & & & \\ P & & & & \end{array}$$

For example, every object of $\underline{\mathrm{BMod}}_A$ is a quotient of a **topologically free** A-module (a completion of a free module). Hence:

- The category $\underline{\mathrm{BMod}}_A$ has enough projectives.
- An object of $\underline{\mathrm{BMod}}_A$ is projective iff it is a summand of a topologically free module.

Let $\underline{\mathrm{BPMod}}_A$ be the full subcategory of projective objects of $\underline{\mathrm{BMod}}_A$.

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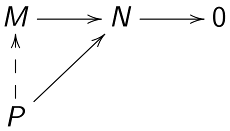

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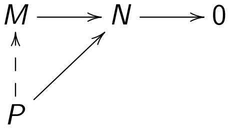

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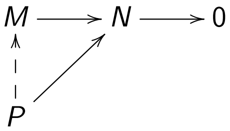

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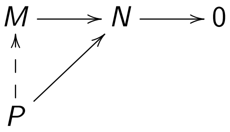

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Sheafy Huber rings

One difficulty of Huber's theory of adic spaces is that, for A a complete Tate Huber ring (and A^+ a ring of integral elements), the adic spectrum $\mathrm{Spa}(A, A^+)$ carries a natural **structure presheaf** which is **not** necessarily a sheaf.

Concretely, this means that for $f, g \in A$ generating the unit ideal, the Čech sequence

$$0 \to A \to \frac{A\langle T\rangle}{\overline{(f - Tg)}} \oplus \frac{A\langle U\rangle}{\overline{(g - Uf)}} \to \frac{A\langle T, U\rangle}{\overline{(f - Tg, g - Uf, TU - 1)}} \to 0$$

is not necessarily exact. (Note the closures of ideals.)

We say that A is **sheafy** if the structure presheaf $\mathcal{O}$ on $\mathrm{Spa}(A, A^+)$ is a sheaf. In this case, $H^i(\mathrm{Spa}(A, A^+), \mathcal{O}) = 0$ for all $i > 0$, so the above sequence is exact; moreover, the closures are not required (KSK–Liu).

Banach bundles and Banach sheaves

Let A be a sheafy complete Tate Huber ring and put $X = \mathrm{Spa}(A, A^+)$ (for some choice of A^+). For every $M \in \underline{\mathrm{BMod}}_A$, we can define a presheaf $\tilde{M} = M \widehat{\otimes}_A \mathcal{O}$ on X. A **Banach bundle** is a sheaf on X which is locally isomorphic to the presheaf associated to some **projective** Banach module in this way.

Theorem (KSK; after Tate, Kiehl, KSK–Liu)

The category $\underline{\mathrm{BPMod}}_A$ is equivalent to the category of Banach bundles on X (via the global sections functor). Moreover, for any such sheaf $\mathcal{F}$, $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Aside for experts: Banach bundles are also acyclic sheaves for the étale topology, the pro-étale topology, and the v-topology.

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Warning: here be dragons!

In general, it is quite difficult to decide whether a given Huber ring is sheafy. Examples include affinoid algebras (Tate) and **perfectoid rings** (KSK–Liu, Scholze). It is a question of active interest to develop useful criteria for sheafiness (e.g., Hansen–KSK).

Even when A is sheafy, without some noetherian hypothesis, a localization map $A \to B$ can fail to be flat. However, they are in some sense "topologically flat" (KSK–Liu).

One way to circumvent these issues is to work in a suitably "derived" framework (Bambozzi–Kremnitzer, Clausen–Scholze). Here we limit ourselves to situations where we can avoid this; our results should embed into such a treatment as a special case where everything is "acyclic".

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Reminder: adic and schematic FF curves

Let F be a perfect field of characteristic p complete for a nontrivial absolute value.[¶] As in part 1, define the **adic FF curve** X_F^{an} and the **schematic FF curve** X_F. Reminder: Y_F is the subspace of $\mathrm{Spa}(W(\mathfrak{o}_F), W(\mathfrak{o}_F))$ obtained by omitting the fixed points of φ; X_F^{an} is the quotient of Y_F by the (properly discontinuous) action of φ; and

$$X_F = \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} \Gamma(X_F^{\mathrm{an}}, \mathcal{O}(n))\right)$$

where $\mathcal{O}(n)$ is the line bundle corresponding to the trivial line bundle on Y_F with the action of φ on a free generator being multiplication by p^{-n}.

[¶] That is, F is a **perfectoid field** of characteristic p.

Products of FF curves

Let d be a positive integer (the case $d = 1$ will reproduce the previous slide). Let $(X_F^d)^{\mathrm{an}}$ be the d-fold product of X_F^{an} over $\mathbb{Q}_p$. This is a **sousperfectoid** space (Hansen–KSK), hence a genuine adic space. It is **not** locally noetherian, so we do not try to study coherent sheaves on it.

Define also the scheme

$$X_F^d = \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} \Gamma((X_F^d)^{\mathrm{an}}, \mathcal{O}(n) \boxtimes \cdots \boxtimes \mathcal{O}(n))\right)$$

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The main theorem

Theorem (KSK)

Pullback along the morphism $(X_F^d)^{\mathrm{an}} \to X_F^d$ of locally ringed spaces induces an equivalence of categories for vector bundles. Moreover, sheaf cohomology groups are preserved.

Open question: can one define a meaningful notion of "finite type" (in quotes!!) that would include closed analytic subspaces of $(X_F^d)^{\mathrm{an}}$? (Remember, X_F^{an} is not itself topologically of finite type over any field.)

The proof of this will require use of Banach bundles!

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Relative Fargues–Fontaine curves

Let S be a perfectoid space of characteristic p (e.g., $\mathrm{Spa}(F, F^\circ)$).

Let Y_S be the subspace of $\mathrm{Spa}(W(S^\circ), W(S^\circ))$ obtained by removing the fixed points of the Frobenius map $\varphi : W(S^\circ) \to W(S^\circ)$. Then φ induces a properly discontinuous automorphism of Y_S; the **analytic relative Fargues–Fontaine curve** is $X_S^{\mathrm{an}} = Y_S/\varphi$.

The **schematic relative Fargues–Fontaine curve** is the scheme

$$X_S = \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} \Gamma(X_S^{\mathrm{an}}, \mathcal{O}(n))\right)$$

where again $\mathcal{O}(n)$ is the line bundle corresponding to the trivial line bundle on Y_S with the action of φ on a free generator being multiplication by p^{-n}.

Warning: This construction does "lie over S" but only in a quite subtle way (e.g., in Scholze's theory of **diamonds**).

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Comparison with the $d = 1$ case

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Theorem

Pullback along the morphism $(X_F^d)^{\mathrm{an}} \to X_F^d$ of locally ringed spaces induces an equivalence of categories for vector bundles. Moreover, sheaf cohomology groups are preserved.

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For any vector bundle $\mathcal{F}$ on $(X_F^d)^{\mathrm{an}}$, for all $n \gg 0$,

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Now given a Banach bundle $\mathcal{F}$ on $(X_S^d)^{\mathrm{an}}$, we'd like to "push forward" some twist to obtain a Banach bundle on $(X_S^{d-1})^{\mathrm{an}}$; **but** the map $X_S^{\mathrm{an}} \to S$ is only topological. However, it does admit a genuine section and we can **pull back** (no quotes) along (some thickening of) that section.

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