Tetrahedra with rational dihedral angles

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These slides can be downloaded from https://kskedlaya.org/slides/.
Associated code available from https://github.com/kedlaya/tetrahedra/.

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I acknowledge that my workplace occupies unceded ancestral land of the Kumeyaay Nation.

Contents

The main result

Throughout this talk, angles will be measured in radians, and a **rational** angle means one whose measure is a rational multiple of π.

Theorem (conjecture of PR, 1995; theorem of KKPR, 2020)

A tetrahedron has all dihedral angles rational if and only if it appears in one of 2 one-parameter families or a list of 59 sporadic cases (details below).

This is included in a stronger result:

Theorem (theorem of KKPR, 2020)

A configuration of lines through the origin in $\mathbb{R}^3$ has the property that any two of the lines form a rational angle if and only if (classification to be described below).

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Why tetrahedra with rational dihedral angles?

It is an open problem to decide which tetrahedra can be used to tile $\mathbb{R}^3$. Some important partial results:

- Regiomontanus, 1400s (refuting a claim of Aristotle): one cannot tile $\mathbb{R}^3$ with **regular** tetrahedra.
- Debrunner, 1980: Any tetrahedron which tiles[1] $\mathbb{R}^3$ is **rectifiable** = scissors-congruent to a cube (or equivalently to a parallelepiped).
- Dehn, 1903: For any rectifiable tetrahedron, the **Dehn invariant**

$$\sum_{jk} \ell_{jk} \otimes \alpha_{jk} \in \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{R}/\pi\mathbb{Q}$$

  is zero. Here ℓ_{jk} denotes the length of the edge between vertices j and k, and α_{jk} denotes the dihedral angle along that edge.
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More on rectifiable tetrahedra

Many examples of rectifiable tetrahedra are known, but a complete classification seems intractable.[2]

However, Conway–Jones (1974) observed that tetrahedra with rational dihedral angles form a subclass of rectifiable tetrahedra which **can** in principle be classified; in particular they reduced this to a finite (but infeasible) computation.

Our contribution can be divided into three parts.

1. Further reduce this finite computation to a larger collection of smaller computations.
2. Use computer algebra to reduce **these** computations to a single large **numerical** computation.
3. Do the numerical computation rigorously.

[2] Our work handles the case where the angles span a 0-dimensional subspace of $\mathbb{R}/\pi\mathbb{Q}$. Chentouf–Poonen–Sun handle the case where the span has dimension ≥ 5.

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Contents

The diagonals of a regular polygon

In 1994, Poonen and Rubinstein[3] did a summer internship at Bell Labs. During this time, they solved[4] the following problem: for n a positive integer, express the number of interior points of a regular n-gon lying on more than one diagonal **in closed form** in n. The main term is $\binom{n}{4}$, with a correction from cases where three or more diagonals concur at a point.

Key point: every three-way intersection corresponds to a solution of

$$\cos\theta_1 + \cdots + \cos\theta_6 = 0, \qquad \theta_1, \ldots, \theta_6 \in 2\pi\mathbb{Q}.$$

We thus reduce to classifying 12-term additive relations among elements of $\mu \subseteq \mathbb{C}^\times$, the group of roots of unity, i.e., solutions of

$$z_1 + \cdots + z_{12} = 0, \qquad z_1, \ldots, z_{12} \in \mu.$$

[3] Poonen had just finished his PhD; Rubinstein finished his in 1998.

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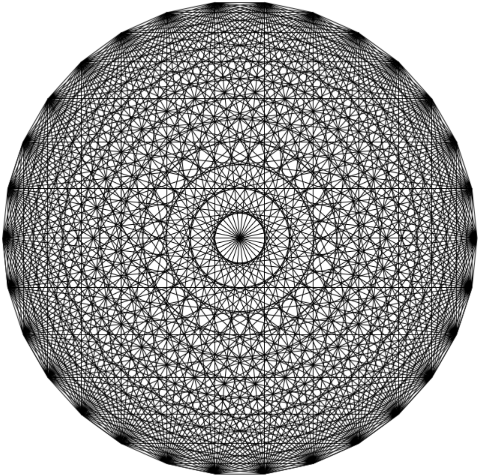
The case $n = 30$ (illustration from Poonen–Rubenstein)



The 30-gon with its diagonals. There are 16801 interior intersection points: 13800 two line intersections, 2250 three line intersections, 420 four line intersections, 180 five line intersections, 120 six line intersections, 30 seven line intersections, and 1 fifteen line intersection. [Note: $\binom{30}{4} = 27405$.]

Classifying sums of roots of unity

It is natural to rephrase in terms of classifying solutions of

$$z_1 + \cdots + z_n = 0, \qquad z_1, \dots, z_n \in \mu$$

which are **minimal**[5], i.e., no nonempty proper subset sums to zero (modulo the symmetries of permutation and scalar multiplication).

For example, for $n = 2$, we must have $z_2 = -z_1$. For $n = 3$, we must have

$$(z_1, z_2, z_3) \sim (1, \zeta_3, \zeta_3^2) \qquad (\zeta_3 = e^{2\pi i/3}).$$

For $n = 4$, there are no minimal relations. For $n = 5$, we must have

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For $n = 6$, things start to get interesting: we must have

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Classifying sums of roots of unity (continued)

It is known that for any fixed n, the classification of additive relations among at most n roots of unity reduces to a finite computation (described by Conway–Jones using ideas of Cassels). This has been carried out in these cases:

- $n \leq 7$: Mann, 1965.
- $n \leq 8$: Włodarski, 1969.
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While doing a literature search, Poonen–Rubinstein found the Conway–Jones paper and its proposal to identify tetrahedra with rational dihedral angles. Game on!

[6] These authors were motivated by applications in combinatorics, of which there are many. Please let me know your favorites!

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The Gram determinant

We can classify tetrahedra with rational dihedral angles by classifying 4-tuples of unit vectors in $\mathbb{R}^3$ which form all rational angles.

Here we follow a suggestion[7] of Igor Rivin. Let M be the 3×4 matrix with these column vectors; then

$$A = M^T M = \begin{pmatrix} 1 & \cos\theta_{12} & \cos\theta_{13} & \cos\theta_{14} \\ \cos\theta_{12} & 1 & \cos\theta_{23} & \cos\theta_{24} \\ \cos\theta_{13} & \cos\theta_{23} & 1 & \cos\theta_{34} \\ \cos\theta_{14} & \cos\theta_{24} & \cos\theta_{34} & 1 \end{pmatrix}$$

has rank at most 3, so its determinant is 0. Rescaling, the equation $\det(2A) = 0$ is a polynomial in the six variables $z_{jk} = e^{i\theta_{jk}}$, which we want to take values which are roots of unity.

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Numerical computations and a conjecture

By computing examples with $z_{jk} \in \mu_N$ for $N \leq 210$, Poonen–Rubinstein arrived at the following conjecture, which we prove in our work.

Theorem (conjecture of PR, 1995; theorem of KKPR, 2020)

Up to symmetry, any tetrahedron in $\mathbb{R}^3$ with all dihedral angles rational is either one of 59 sporadic examples (next slide) or has one of the forms

$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi - 2x, \frac{\pi}{3}, x, x\right) \qquad \text{for } \frac{\pi}{6} < x < \frac{\pi}{2},$$
$$\left(\frac{5\pi}{6} - x, \frac{\pi}{6} + x, \frac{2\pi}{3} - x, \frac{2\pi}{3} - x, x, x\right) \qquad \text{for } \frac{\pi}{6} < x \leq \frac{\pi}{3}.$$

The first parametric example was discovered by Hill in 1895. The second example was discovered by Poonen–Rubinstein, but turns out to be related to the first via a symmetry **not** generated by S_4; more on this later.

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$$\left(\frac{\pi}{2}, \frac{\pi}{2}, \pi - 2x, \frac{\pi}{3}, x, x\right) \qquad \text{for } \frac{\pi}{6} < x < \frac{\pi}{2},$$

$$\left(\frac{5\pi}{6} - x, \frac{\pi}{6} + x, \frac{2\pi}{3} - x, \frac{2\pi}{3} - x, x, x\right) \qquad \text{for } \frac{\pi}{6} < x \leq \frac{\pi}{3}.$$

The first parametric example was discovered by Hill in 1895. The second example was discovered by Poonen–Rubinstein, but turns out to be related to the first via a symmetry **not** generated by S_4; more on this later.

Numerical computations and a conjecture

By computing examples with $z_{jk} \in \mu_N$ for $N \leq 210$, Poonen–Rubinstein arrived at the following conjecture, which we prove in our work.

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Sporadic tetrahedra (key on the next slide)

N	$(\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23})$ as multiples of π/N
12	$(3, 4, 3, 4, 6, 8) = H_2(\pi/4)$
24	$(5, 9, 6, 8, 13, 15)$
12	$(3, 6, 4, 6, 4, 6) = T_0$
24	$(7, 11, 7, 13, 8, 12)$
15	$(3, 3, 3, 5, 10, 10) = T_{18}$, $(2, 4, 4, 4, 10, 10)$, $(3, 3, 4, 4, 9, 11)$
15	$(3, 3, 5, 5, 9, 9) = T_7$
15	$(5, 5, 5, 9, 6, 6) = T_{23}$, $(3, 7, 6, 6, 7, 7)$, $(4, 8, 5, 5, 7, 7)$
21	$(3, 9, 7, 7, 12, 12)$, $(4, 10, 6, 6, 12, 12)$, $(6, 6, 7, 7, 9, 15)$
30	$(6, 12, 10, 15, 10, 20) = T_{17}$, $(4, 14, 10, 15, 12, 18)$
60	$(8, 28, 19, 31, 25, 35)$, $(12, 24, 15, 35, 25, 35)$, $(13, 23, 15, 35, 24, 36)$, $(13, 23, 19, 31, 20, 40)$
30	$(6, 18, 10, 10, 15, 15) = T_{13}$, $(4, 16, 12, 12, 15, 15)$, $(9, 21, 10, 10, 12, 12)$
30	$(6, 6, 10, 12, 15, 20) = T_{16}$, $(5, 7, 11, 11, 15, 20)$
60	$(7, 17, 20, 24, 35, 35)$, $(7, 17, 22, 22, 33, 37)$, $(10, 14, 17, 27, 35, 35)$, $(12, 12, 17, 27, 33, 37)$
30	$(6, 10, 10, 15, 12, 18) = T_{21}$, $(5, 11, 10, 15, 13, 17)$
60	$(10, 22, 21, 29, 25, 35)$, $(11, 21, 19, 31, 26, 34)$, $(11, 21, 21, 29, 24, 36)$, $(12, 20, 19, 31, 25, 35)$
30	$(6, 10, 6, 10, 15, 24) = T_6$
60	$(7, 25, 12, 20, 35, 43)$
30	$(6, 12, 6, 12, 15, 20) = T_2$
60	$(12, 24, 13, 23, 29, 41)$
30	$(6, 12, 10, 10, 15, 18) = T_3$, $(7, 13, 9, 9, 15, 18)$
60	$(12, 24, 17, 23, 33, 33)$, $(14, 26, 15, 21, 33, 33)$, $(15, 21, 20, 20, 27, 39)$, $(17, 23, 18, 18, 27, 39)$
30	$(6, 15, 6, 18, 10, 20) = T_4$, $(6, 15, 7, 17, 9, 21)$
60	$(9, 33, 14, 34, 21, 39)$, $(9, 33, 15, 33, 20, 40)$, $(11, 31, 12, 36, 21, 39)$, $(11, 31, 15, 33, 18, 42)$
30	$(6, 15, 10, 15, 12, 15) = T_1$, $(6, 15, 11, 14, 11, 16)$, $(8, 13, 8, 17, 12, 15)$, $(8, 13, 9, 18, 11, 14)$, $(8, 17, 9, 12, 11, 16)$, $(9, 12, 9, 18, 10, 15)$
30	$(10, 12, 10, 12, 15, 12) = T_5$
60	$(19, 25, 20, 24, 29, 25)$

How to read the table

Each tetrahedron is represented by an integer N and a list of six integers, representing the dihedral angles $\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23}$ as multiples of $\frac{\pi}{N}$. (Here α_{jk} means the angle between faces j and k.)

The extra labels indicate examples of tetrahedra that we found in the literature as examples of rectifiable tetrahedra. All of these come from 4-line configurations within the maximal 9-line and 15-line configurations.

The groups between horizontal lines are orbits for the "big" symmetry group (more on this later).

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Bad news and good news

The bad news: we would like to solve by treating each monomial in $\det(2A) = 0$ as a separate root of unity and classifying additive relations. But there are 105 monomials, far beyond the feasible range.

The good news: the equation reduces modulo 2 to an equation with only 12 terms! Fortunately, one can also classify additive relations modulo 2 in a similar manner: every minimal mod-2 additive relation is the reduction of a genuine additive relation.[8]

Fitting the classification against the mod-2 polynomial, we obtain a bunch of solutions in up to 3 parameters. What then remains is to find the solutions of the original equation.

[8] Using the Christie–Dykema–Klep work, we showed with Toren D'Nelly-Warady that this persist up to $n \leq 18$ but fails for $n = 19$: there is a minimal relation of length 21 with a repeated summand.

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Halftime!

This problem was proposed for the 2005 Duluth REU. Jack Huizenga worked on it but concluded that it was infeasible at the time.

A forgotten symmetry

In 1968, the physicists Ponzano and Regge made a remarkable discovery about the geometry of tetrahedra. This seems to have gone unnoticed by mathematicians until the work of Justin Roberts in 1999.

Theorem (Ponzano–Regge, 1968; Roberts, 1999)

For any tetrahedron with edge lengths ℓ_{jk} and dihedral angles α_{jk}, set

$$s := \frac{1}{2}(\ell_{13} + \ell_{24} + \ell_{14} + \ell_{23}), \qquad x = \frac{1}{2}(\alpha_{13} + \alpha_{24} + \alpha_{14} + \alpha_{23}).$$

Then the tuples

$$(\ell_{12}, \ell_{34}, s - \ell_{13}, s - \ell_{24}, s - \ell_{14}, s - \ell_{23}),$$
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Effect of the Regge symmetry

The Gram equation $\det(2A) = 0$ has an obvious symmetry by S_4, but adding the Regge symmetry gives the much larger group $W(D_6)$ of order $2^5 6! = 23040$.

This implies a significant simplification in our approach. For instance, of the mod-2 solutions, there are initially 5760 three-parameter families, but the $W(D_6)$-action reduces this to 14.

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More on equations in roots of unity

By restricting the Gram equation to parametric solutions of the mod-2 equation, we obtain a few hundred subproblems of the form (with $n \leq 3$): for some polynomial $P(z_1, \ldots, z_n)$ over $\mathbb{Q}(\zeta_m)$ for some m, find all solutions of $P(z_1, \ldots, z_n) = 0$ with $z_1, \ldots, z_n \in \mu$. These polynomials are not necessarily sparse, so the Conway–Jones approach is not always applicable.

This problem is easy when $n = 1$. It turns out there is an efficient approach for $n > 1$ which is very practical for $n = 2$ and workable for $n = 3$ (but as yet not for $n \geq 4$).

Aside: this is closely related to effective approaches to finding torsion points on algebraic curves (as in the Manin–Mumford conjecture).

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A key example: the algorithm of Beukers–Smyth

Lemma

For $P(x, y) \in \mathbb{Z}[x, y]$, every solution of $P(x, y) = 0$ with $x, y \in \mu$ is also a solution of one of

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Proof sketch: the groups $\langle x, y \rangle / \langle x \rangle, \langle x, y \rangle / \langle y \rangle$ are finite and their orders are not both even. Split into cases based on these orders mod 4.

As long as P is not of the form $x^i y^j Q(x^2, y^2)$, each solution ends up in a system of two equations with only finitely many solutions over $\mathbb{C}$. It is easy to pick out the solutions in roots of unity.

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Using computer algebra (mostly SageMath), we find a collection of (sometimes parametric) solutions in roots of unity to the Gram equation.

We can check by inspection that the parametric solutions correspond to exactly the known ones.

As for the isolated solutions, it is easiest to just check that they all involve N-th roots of unity for some $N \leq 420$, and rerun[9] the numerical computation to find all sporadic solutions up to that point.

Note: we also end up classifying **degenerate** solutions to the Gram equation. This yields an extra result...

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A classification theorem

By a **rational-angle line configuration**, I mean a set of lines in $\mathbb{R}^3$ through the origin such that any two form a rational angle.

Theorem (KKPR, 2020)

The maximal rational-angle line configurations are classified as in the following table.

n	number of maximal rational-angle n-line configurations
$\aleph_0$	1
15	1
9	1
8	5
6	22, plus 5 one-parameter families
5	29, plus 2 one-parameter families
4	228, plus 10 one-parameter families and 2 two-parameter families
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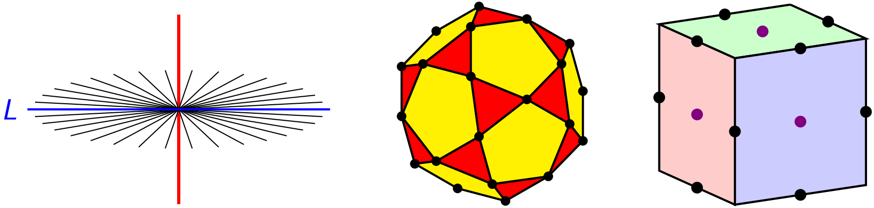
Three maximal configurations

The maximal configurations of size $\aleph_0$, 15, 9 can be depicted as follows.



The left figure consists of a family of lines perpendicular to the red line. In the other two figures, one draws the diameters through the marked points, in either the **icosidodecahedron** or the **B3 root system**.

There are 5 different 8-line configurations consisting of seven central diagonals of a regular 60-gon, plus an eighth line not in the same plane.

Happy birthday Joe!