Uniformities for F-isocrystals on curves

Kiran S. Kedlaya

Department of Mathematics, University of California San Diego
kedlaya@ucsd.edu

These slides can be downloaded from https://kskedlaya.org/slides/.

Dwork seminar
April 5, 2023

Supported by NSF (grant DMS-2053473 and previous grants), UC San Diego (Warschawski Professorship), and the Society of p-adicts.

The UC San Diego campus sits on unceded ancestral land of the Kumeyaay Nation.

Contents

Basic setup

Let k be a perfect field of characteristic $p > 0$. We sometimes assume that k is finite, in which case $q := \#k$.

Let $\overline{X}$ be a smooth, projective, geometrically irreducible curve over k. Let X be a nonempty open affine subscheme of $\overline{X}$. Let Z be the (reduced) complement of X in $\overline{X}$.

Let g be the genus of $\overline{X}$. Let m be the k-length of Z.

Basic setup

Let k be a perfect field of characteristic $p > 0$. We sometimes assume that k is finite, in which case $q := \#k$.

Let $\overline{X}$ be a smooth, projective, geometrically irreducible curve over k. Let X be a nonempty open affine subscheme of $\overline{X}$. Let Z be the (reduced) complement of X in $\overline{X}$.

Let g be the genus of $\overline{X}$. Let m be the k-length of Z.

Basic setup

Let k be a perfect field of characteristic $p > 0$. We sometimes assume that k is finite, in which case $q := \#k$.

Let $\overline{X}$ be a smooth, projective, geometrically irreducible curve over k. Let X be a nonempty open affine subscheme of $\overline{X}$. Let Z be the (reduced) complement of X in $\overline{X}$.

Let g be the genus of $\overline{X}$. Let m be the k-length of Z.

Lifts to characteristic 0

Let $\overline{\mathfrak{X}}$ be[1] a smooth projective scheme over $W(k)$ (the Witt vectors) equipped with an isomorphism $\overline{\mathfrak{X}}_k \cong \overline{X}$.

Let $\mathfrak{Z}$ be a smooth divisor in $\overline{\mathfrak{X}}$ such that $\mathfrak{Z}_k \subset \overline{\mathfrak{X}}_k$ is identified with $Z \subset \overline{X}$.

Let K be the fraction field of $W(k)$. Let $\overline{\mathfrak{X}}_K^{\mathrm{an}}$ be the analytification[2] of $\overline{\mathfrak{X}}_K$. We may then view $\overline{\mathfrak{Z}}_K$ as a closed analytic subspace of $\overline{\mathfrak{X}}_K^{\mathrm{an}}$.

[1] Reminder: such a lift always exists because of the smoothness of the moduli stack of curves.

[2] For this talk, it is immaterial whether this is done in the category of rigid analytic spaces, Berkovich analytic spaces, or Huber adic spaces.

Lifts to characteristic 0

Let $\overline{\mathfrak{X}}$ be[1] a smooth projective scheme over $W(k)$ (the Witt vectors) equipped with an isomorphism $\overline{\mathfrak{X}}_k \cong \overline{X}$.

Let $\mathfrak{Z}$ be a smooth divisor in $\overline{\mathfrak{X}}$ such that $\mathfrak{Z}_k \subset \overline{\mathfrak{X}}_k$ is identified with $Z \subset \overline{X}$.

Let K be the fraction field of $W(k)$. Let $\overline{\mathfrak{X}}_K^{\mathrm{an}}$ be the analytification[2] of $\overline{\mathfrak{X}}_K$. We may then view $\overline{\mathfrak{Z}}_K$ as a closed analytic subspace of $\overline{\mathfrak{X}}_K^{\mathrm{an}}$.

[1] Reminder: such a lift always exists because of the smoothness of the moduli stack of curves.

[2] For this talk, it is immaterial whether this is done in the category of rigid analytic spaces, Berkovich analytic spaces, or Huber adic spaces.

Lifts to characteristic 0

Let $\overline{\mathfrak{X}}$ be[1] a smooth projective scheme over $W(k)$ (the Witt vectors) equipped with an isomorphism $\overline{\mathfrak{X}}_k \cong \overline{X}$.

Let $\mathfrak{Z}$ be a smooth divisor in $\overline{\mathfrak{X}}$ such that $\mathfrak{Z}_k \subset \overline{\mathfrak{X}}_k$ is identified with $Z \subset \overline{X}$.

Let K be the fraction field of $W(k)$. Let $\overline{\mathfrak{X}}_K^{\mathrm{an}}$ be the analytification[2] of $\overline{\mathfrak{X}}_K$. We may then view $\overline{\mathfrak{Z}}_K$ as a closed analytic subspace of $\overline{\mathfrak{X}}_K^{\mathrm{an}}$.

[1] Reminder: such a lift always exists because of the smoothness of the moduli stack of curves.

[2] For this talk, it is immaterial whether this is done in the category of rigid analytic spaces, Berkovich analytic spaces, or Huber adic spaces.

Strict neighborhoods

We may also view $\overline{\mathfrak{X}}_K^{\mathrm{an}}$ as the Raynaud generic fiber of the formal completion of $\overline{\mathfrak{X}}$ along $\overline{\mathfrak{X}}_k$. Let $\mathfrak{X}$ be the open formal subscheme of the completion supported on $X \subset \overline{X}$, and let $\mathfrak{X}_K^{\mathrm{an}} \subset \overline{\mathfrak{X}}_K^{\mathrm{an}}$ denote the Raynaud generic fiber of $\mathfrak{X}$.

The complement of $\mathfrak{X}_K^{\mathrm{an}}$ in $\overline{\mathfrak{X}}_K^{\mathrm{an}}$ consists of a finite union of virtual[3] open discs. A **strict neighborhood** of $\mathfrak{X}_K^{\mathrm{an}}$ in $\overline{\mathfrak{X}}_K^{\mathrm{an}}$ is an open neighborhood of $\mathfrak{X}_K^{\mathrm{an}}$ whose complement in $\overline{\mathfrak{X}}_K^{\mathrm{an}}$ is contained in some **quasicompact** open subspace of the complement of $\mathfrak{X}_K^{\mathrm{an}}$. That is, one replaces each open disc with some disc of **strictly smaller** radius.

[3] That is, these become open discs after a finite base extension on k, to make the points of Z become k-rational.

Strict neighborhoods

We may also view $\overline{\mathfrak{X}}_K^{\mathrm{an}}$ as the Raynaud generic fiber of the formal completion of $\overline{\mathfrak{X}}$ along $\overline{\mathfrak{X}}_k$. Let $\mathfrak{X}$ be the open formal subscheme of the completion supported on $X \subset \overline{X}$, and let $\mathfrak{X}_K^{\mathrm{an}} \subset \overline{\mathfrak{X}}_K^{\mathrm{an}}$ denote the Raynaud generic fiber of $\mathfrak{X}$.

The complement of $\mathfrak{X}_K^{\mathrm{an}}$ in $\overline{\mathfrak{X}}_K^{\mathrm{an}}$ consists of a finite union of virtual[3] open discs. A **strict neighborhood** of $\mathfrak{X}_K^{\mathrm{an}}$ in $\overline{\mathfrak{X}}_K^{\mathrm{an}}$ is an open neighborhood of $\mathfrak{X}_K^{\mathrm{an}}$ whose complement in $\overline{\mathfrak{X}}_K^{\mathrm{an}}$ is contained in some **quasicompact** open subspace of the complement of $\mathfrak{X}_K^{\mathrm{an}}$. That is, one replaces each open disc with some disc of **strictly smaller** radius.

[3] That is, these become open discs after a finite base extension on k, to make the points of Z become k-rational.

Overconvergent F-isocrystals

Let $\varphi_K : K \to K$ be the Witt vector Frobenius. An **overconvergent F-isocrystal** on X is a vector bundle $\mathcal{E}$ with connection[4] on some[5] strict neighborhood V of $\mathfrak{X}_K^{\mathrm{an}}$ in $\overline{\mathfrak{X}}_K^{\mathrm{an}}$, together with an isomorphism $\varphi_V^* \mathcal{E} \cong \mathcal{E}$ of vector bundles with connection where $\varphi_V : V \to V$ is some φ_K-semilinear map extending an absolute Frobenius lift on $\mathfrak{X}_K^{\mathrm{an}}$.

Let $\mathbf{FIsoc}^\dagger(X)$ be the category of overconvergent F-isocrystals on X (where morphisms respect the connection and Frobenius structure). As the notation suggests, this category is functorial in X; in particular it does depend on the choice of the lift $\overline{\mathfrak{X}}$ of $\overline{X}$.

These objects show up in Berthelot's **rigid cohomology** as the analogue of Weil $\overline{\mathbb{Q}}_\ell$-sheaves in étale cohomology. More on this later.

[4] No integrability condition because $\dim(X) = 1$.

[5] The strict neighborhood is unspecified; that is, the category of overconvergent F-isocrystals is a 2-colimit over choices of the strict neighborhood.

Overconvergent F-isocrystals

Let $\varphi_K : K \to K$ be the Witt vector Frobenius. An **overconvergent F-isocrystal** on X is a vector bundle $\mathcal{E}$ with connection[4] on some[5] strict neighborhood V of $\mathfrak{X}_K^{\mathrm{an}}$ in $\overline{\mathfrak{X}}_K^{\mathrm{an}}$, together with an isomorphism $\varphi_V^* \mathcal{E} \cong \mathcal{E}$ of vector bundles with connection where $\varphi_V : V \to V$ is some φ_K-semilinear map extending an absolute Frobenius lift on $\mathfrak{X}_K^{\mathrm{an}}$.

Let $\mathbf{FIsoc}^\dagger(X)$ be the category of overconvergent F-isocrystals on X (where morphisms respect the connection and Frobenius structure). As the notation suggests, this category is functorial in X; in particular it does depend on the choice of the lift $\overline{\mathfrak{X}}$ of $\overline{X}$.

These objects show up in Berthelot's **rigid cohomology** as the analogue of Weil $\overline{\mathbb{Q}}_\ell$-sheaves in étale cohomology. More on this later.

[4]No integrability condition because $\dim(X) = 1$.

[5]The strict neighborhood is unspecified; that is, the category of overconvergent F-isocrystals is a 2-colimit over choices of the strict neighborhood.

Overconvergent F-isocrystals

Let $\varphi_K : K \to K$ be the Witt vector Frobenius. An **overconvergent F-isocrystal** on X is a vector bundle $\mathcal{E}$ with connection[4] on some[5] strict neighborhood V of $\mathfrak{X}_K^{\mathrm{an}}$ in $\overline{\mathfrak{X}}_K^{\mathrm{an}}$, together with an isomorphism $\varphi_V^* \mathcal{E} \cong \mathcal{E}$ of vector bundles with connection where $\varphi_V : V \to V$ is some φ_K-semilinear map extending an absolute Frobenius lift on $\mathfrak{X}_K^{\mathrm{an}}$.

Let $\mathbf{FIsoc}^\dagger(X)$ be the category of overconvergent F-isocrystals on X (where morphisms respect the connection and Frobenius structure). As the notation suggests, this category is functorial in X; in particular it does depend on the choice of the lift $\overline{\mathfrak{X}}$ of $\overline{X}$.

These objects show up in Berthelot's **rigid cohomology** as the analogue of Weil $\overline{\mathbb{Q}}_\ell$-sheaves in étale cohomology. More on this later.

[4] No integrability condition because $\dim(X) = 1$.

[5] The strict neighborhood is unspecified; that is, the category of overconvergent F-isocrystals is a 2-colimit over choices of the strict neighborhood.

Contents

Local isocrystals

What is the analogue of $\mathbf{FIsoc}^\dagger(X)$ with X replaced by $s = \operatorname{Spec} k((t))$?

The **Robba ring** $\mathcal{R}_K$ is the colimit ($=$ union) of the rings of analytic functions over K on the annulus $\rho < |t| < 1$ as $\rho \to 1^-$. Such functions can be identified with Laurent series $\sum_{n \in \mathbb{Z}} c_n t^n$ with $c_n \in K$ such that

$$\limsup_{n \to -\infty} |c_n| \rho^n < \infty \text{ for } \textbf{some } \rho \in (0,1)$$
$$\limsup_{n \to +\infty} |c_n| \rho^n < \infty \text{ for } \textbf{all } \rho \in (0,1).$$

We take $\mathbf{FIsoc}^\dagger(s)$ to be the category of finite free[6] modules over $\mathcal{R}_K$ equipped with compatible actions of the derivation $\frac{d}{dt}$ and some Frobenius lift φ on $\mathcal{R}_K$. Again, this implies the same for any other choice of φ.

For $x \in Z(k)$, we get a pullback functor $\mathbf{FIsoc}^\dagger(X) \to \mathbf{FIsoc}^\dagger(s)$.

[6] Any finite projective module over $\mathcal{R}_K$ is in fact free.

Local isocrystals

What is the analogue of $\mathbf{FIsoc}^\dagger(X)$ with X replaced by $s = \operatorname{Spec} k((t))$?

The **Robba ring** $\mathcal{R}_K$ is the colimit (= union) of the rings of analytic functions over K on the annulus $\rho < |t| < 1$ as $\rho \to 1^-$. Such functions can be identified with Laurent series $\sum_{n \in \mathbb{Z}} c_n t^n$ with $c_n \in K$ such that

$$\limsup_{n \to -\infty} |c_n| \rho^n < \infty \text{ for } \textbf{some } \rho \in (0,1)$$

$$\limsup_{n \to +\infty} |c_n| \rho^n < \infty \text{ for } \textbf{all } \rho \in (0,1).$$

We take $\mathbf{FIsoc}^\dagger(s)$ to be the category of finite free[6] modules over $\mathcal{R}_K$ equipped with compatible actions of the derivation $\frac{d}{dt}$ and some Frobenius lift φ on $\mathcal{R}_K$. Again, this implies the same for any other choice of φ.

For $x \in Z(k)$, we get a pullback functor $\mathbf{FIsoc}^\dagger(X) \to \mathbf{FIsoc}^\dagger(s)$.

[6] Any finite projective module over $\mathcal{R}_K$ is in fact free.

Local isocrystals

What is the analogue of $\mathbf{FIsoc}^\dagger(X)$ with X replaced by $s = \operatorname{Spec} k((t))$?

The **Robba ring** $\mathcal{R}_K$ is the colimit (= union) of the rings of analytic functions over K on the annulus $\rho < |t| < 1$ as $\rho \to 1^-$. Such functions can be identified with Laurent series $\sum_{n\in\mathbb{Z}} c_n t^n$ with $c_n \in K$ such that

$$\limsup_{n\to-\infty} |c_n|\rho^n < \infty \text{ for } \textbf{some } \rho \in (0,1)$$
$$\limsup_{n\to+\infty} |c_n|\rho^n < \infty \text{ for } \textbf{all } \rho \in (0,1).$$

We take $\mathbf{FIsoc}^\dagger(s)$ to be the category of finite free[6] modules over $\mathcal{R}_K$ equipped with compatible actions of the derivation $\frac{d}{dt}$ and some Frobenius lift φ on $\mathcal{R}_K$. Again, this implies the same for any other choice of φ.

For $x \in Z(k)$, we get a pullback functor $\mathbf{FIsoc}^\dagger(X) \to \mathbf{FIsoc}^\dagger(s)$.

[6] Any finite projective module over $\mathcal{R}_K$ is in fact free.

Local isocrystals

What is the analogue of $\mathbf{FIsoc}^\dagger(X)$ with X replaced by $s = \operatorname{Spec} k((t))$?

The **Robba ring** $\mathcal{R}_K$ is the colimit ($=$ union) of the rings of analytic functions over K on the annulus $\rho < |t| < 1$ as $\rho \to 1^-$. Such functions can be identified with Laurent series $\sum_{n \in \mathbb{Z}} c_n t^n$ with $c_n \in K$ such that

$$\limsup_{n \to -\infty} |c_n| \rho^n < \infty \text{ for } \textbf{some } \rho \in (0,1)$$
$$\limsup_{n \to +\infty} |c_n| \rho^n < \infty \text{ for } \textbf{all } \rho \in (0,1).$$

We take $\mathbf{FIsoc}^\dagger(s)$ to be the category of finite free[6] modules over $\mathcal{R}_K$ equipped with compatible actions of the derivation $\frac{d}{dt}$ and some Frobenius lift φ on $\mathcal{R}_K$. Again, this implies the same for any other choice of φ.

For $x \in Z(k)$, we get a pullback functor $\mathbf{FIsoc}^\dagger(X) \to \mathbf{FIsoc}^\dagger(s)$.

[6] Any finite projective module over $\mathcal{R}_K$ is in fact free.

Tame local monodromy

Given $\mathcal{E} \in \mathbf{FIsoc}^\dagger(s)$, for $\rho \in (0,1)$ sufficiently large we can restrict to the disc $|t - t_\rho| < \rho$ where t_ρ is a generic point[7] with $|t_\rho| = \rho$.

Theorem (Christol-Mebkhout, late 1990s)

There exists $b = b(\mathcal{E}) \in \mathbb{Q}_{\geq 0}$ such that as $\rho \to 1^-$, the restriction of $\mathcal{E}$ to $|t - t_\rho| < \rho$ has a basis of horizontal sections on $|t - t_\rho| < \rho^{1+c}$ iff $c \leq b$.

We say $\mathcal{E}$ is **tame** if $b(\mathcal{E}) = 0$. (In older literature, $\mathcal{E}$ satisfies the **Robba condition**.)

[7] In rigid geometry, generic points only appear after a large base extension. They occur more naturally in Berkovich and Huber spaces.

Tame local monodromy

Given $\mathcal{E} \in \mathbf{FIsoc}^\dagger(s)$, for $\rho \in (0,1)$ sufficiently large we can restrict to the disc $|t - t_\rho| < \rho$ where t_ρ is a generic point[7] with $|t_\rho| = \rho$.

Theorem (Christol-Mebkhout, late 1990s)

There exists $b = b(\mathcal{E}) \in \mathbb{Q}_{\geq 0}$ such that as $\rho \to 1^-$, the restriction of $\mathcal{E}$ to $|t - t_\rho| < \rho$ has a basis of horizontal sections on $|t - t_\rho| < \rho^{1+c}$ iff $c \leq b$.

We say $\mathcal{E}$ is **tame** if $b(\mathcal{E}) = 0$. (In older literature, $\mathcal{E}$ satisfies the **Robba condition**.)

[7] In rigid geometry, generic points only appear after a large base extension. They occur more naturally in Berkovich and Huber spaces.

Tame local monodromy

Given $\mathcal{E} \in \mathbf{FIsoc}^\dagger(s)$, for $\rho \in (0,1)$ sufficiently large we can restrict to the disc $|t - t_\rho| < \rho$ where t_ρ is a generic point[7] with $|t_\rho| = \rho$.

Theorem (Christol-Mebkhout, late 1990s)

There exists $b = b(\mathcal{E}) \in \mathbb{Q}_{\geq 0}$ such that as $\rho \to 1^-$, the restriction of $\mathcal{E}$ to $|t - t_\rho| < \rho$ has a basis of horizontal sections on $|t - t_\rho| < \rho^{1+c}$ iff $c \leq b$.

We say $\mathcal{E}$ is **tame** if $b(\mathcal{E}) = 0$. (In older literature, $\mathcal{E}$ satisfies the **Robba condition**.)

[7] In rigid geometry, generic points only appear after a large base extension. They occur more naturally in Berkovich and Huber spaces.

The p-adic local monodromy theorem

The **bounded** germs in $\mathcal{R}_K$ form a two-dimensional local field with residue field $k((t))$, which is incomplete but henselian. Hence finite separable extensions of $k((t))$ canonically induce finite extensions of $\mathcal{R}_K$.

Theorem (André, K, Mebkhout, early 2000s)

For any $\mathcal{E} \in \mathbf{FIsoc}^\dagger(s)$, the pullback of $\mathcal{E}$ along some finite separable extension of $k((t))$ is tame.

This corresponds roughly to Grothendieck's local monodromy theorem for étale lisse $\overline{\mathbb{Q}}_\ell$-sheaves.

The p-adic local monodromy theorem

The **bounded** germs in $\mathcal{R}_K$ form a two-dimensional local field with residue field $k((t))$, which is incomplete but henselian. Hence finite separable extensions of $k((t))$ canonically induce finite extensions of $\mathcal{R}_K$.

Theorem (André, K, Mebkhout, early 2000s)

For any $\mathcal{E} \in \mathbf{FIsoc}^\dagger(s)$, the pullback of $\mathcal{E}$ along some finite separable extension of $k((t))$ is tame.

This corresponds roughly to Grothendieck's local monodromy theorem for étale lisse $\overline{\mathbb{Q}}_\ell$-sheaves.

The p-adic local monodromy theorem

The **bounded** germs in $\mathcal{R}_K$ form a two-dimensional local field with residue field $k((t))$, which is incomplete but henselian. Hence finite separable extensions of $k((t))$ canonically induce finite extensions of $\mathcal{R}_K$.

Theorem (André, K, Mebkhout, early 2000s)

For any $\mathcal{E} \in \mathbf{FIsoc}^\dagger(s)$, the pullback of $\mathcal{E}$ along some finite separable extension of $k((t))$ is tame.

This corresponds roughly to Grothendieck's local monodromy theorem for étale lisse $\overline{\mathbb{Q}}_\ell$-sheaves.

Local monodromy representations and wild ramification

Using the pLMT, one can associate[8] to $\mathcal{E} \in \mathbf{FIsoc}^\dagger(s)$ a representation

$$\rho_{\mathcal{E}} : \pi_1(s) \to \mathrm{GL}_r(\mathbb{Q}_p), \qquad r = \mathrm{rank}(\mathcal{E}).$$

Theorem (Matsuda, early 2000s)

The highest ramification break of $\rho_{\mathcal{E}}$ equals $b(\mathcal{E})$. In particular, $\mathcal{E}$ is tame if and only if $\rho_{\mathcal{E}}$ is tame.

[8] This construction forgets extensions, so it factors through the semisimplification of $\mathcal{E}$ as a vector bundle with connection.

Local monodromy representations and wild ramification

Using the pLMT, one can associate[8] to $\mathcal{E} \in \mathbf{FIsoc}^\dagger(s)$ a representation

$$\rho_{\mathcal{E}} : \pi_1(s) \to \mathrm{GL}_r(\mathbb{Q}_p), \qquad r = \mathrm{rank}(\mathcal{E}).$$

Theorem (Matsuda, early 2000s)

The highest ramification break of $\rho_{\mathcal{E}}$ equals $b(\mathcal{E})$. In particular, $\mathcal{E}$ is tame if and only if $\rho_{\mathcal{E}}$ is tame.

[8] This construction forgets extensions, so it factors through the semisimplification of $\mathcal{E}$ as a vector bundle with connection.

Contents

Domain of definition controls ramification

Conjecture

Choose $\mathcal{E} \in \mathbf{FIsoc}^\dagger(s)$ of rank r which can be realized as a vector bundle with connection and Frobenius structure on $\rho < |t| < 1$ for some fixed ρ. Then $b(\mathcal{E})$ is bounded by some function of p, r, ρ.

A known special case: if $\mathcal{E}$ extends to a logarithmic connection across the entire disc $|t| < 1$, then $\mathcal{E}$ must be tame (see below for a more precise statement).

Domain of definition controls ramification

Conjecture

Choose $\mathcal{E} \in \mathbf{FIsoc}^{\dagger}(s)$ of rank r which can be realized as a vector bundle with connection and Frobenius structure on $\rho < |t| < 1$ for some fixed ρ. Then $b(\mathcal{E})$ is bounded by some function of p, r, ρ.

A known special case: if $\mathcal{E}$ extends to a logarithmic connection across the entire disc $|t| < 1$, then $\mathcal{E}$ must be tame (see below for a more precise statement).

Ramification controls domain of definition

Conjecture (work in progress)

Any $\mathcal{E} \in \mathbf{FIsoc}^\dagger(s)$ of rank r with $b(\mathcal{E}) = b$ can be realized as a vector bundle with connection and Frobenius structure on $\rho < |t| < 1$ for some ρ depending only on p, r, b. Moreover, $\mathcal{E}$ admits a generating set on which the actions of the connection and Frobenius are bounded in operator norm by a function of p, r, b.

It would also be of interest to identify optimal bounds. However, any bounds at all would imply some improvements in "cut-by-curves" criteria for overconvergence of convergent F-isocrystals (Shiho, Grubb–K–Upton).

Ramification controls domain of definition

Conjecture (work in progress)

Any $\mathcal{E} \in \mathbf{FIsoc}^\dagger(s)$ of rank r with $b(\mathcal{E}) = b$ can be realized as a vector bundle with connection and Frobenius structure on $\rho < |t| < 1$ for some ρ depending only on p, r, b. Moreover, $\mathcal{E}$ admits a generating set on which the actions of the connection and Frobenius are bounded in operator norm by a function of p, r, b.

It would also be of interest to identify optimal bounds. However, any bounds at all would imply some improvements in "cut-by-curves" criteria for overconvergence of convergent F-isocrystals (Shiho, Grubb–K–Upton).

More remarks

Progress on these conjectures may have some applications in mixed characteristic, e.g., to the study of Emerton–Gee–Hellmann's moduli stacks of (φ, Γ)-modules.

In general, uniformity problems for overconvergent F-isocrystals on curves must account for the wild ramification at all points of Z. However, resolution of these conjecture will (probably) allow these problems to be reduced to the case where everything is tame; for this reason, I restrict all subsequent conjectures to the tame setting.

More remarks

Progress on these conjectures may have some applications in mixed characteristic, e.g., to the study of Emerton–Gee–Hellmann's moduli stacks of (φ, Γ)-modules.

In general, uniformity problems for overconvergent F-isocrystals on curves must account for the wild ramification at all points of Z. However, resolution of these conjecture will (probably) allow these problems to be reduced to the case where everything is tame; for this reason, I restrict all subsequent conjectures to the tame setting.

Contents

Logarithmic extensions

We say $\mathcal{E} \in \textbf{FIsoc}^\dagger(X)$ is **tame** if for every $z \in Z$, after making a base extension to ensure that $z \in Z(k)$, the pullback of $\mathcal{E}$ to $\textbf{FIsoc}^\dagger(s)$ defined by z is tame (i.e., its local monodromy representation is tame).

Theorem

$\mathcal{E}$ is tame iff it extends to a vector bundle with logarithmic connection on $(\overline{\mathfrak{X}}_K^{\mathrm{an}}, \mathfrak{Z}_K^{\mathrm{an}})$ with exponents in $\mathbb{Z}_{(p)}$. In this case, there is a unique such extension with exponents in $\mathbb{Z}_{(p)} \cap [0, 1)$.

Beware that we cannot include the Frobenius structure in this statement because in general there is no Frobenius lift on all of $\overline{\mathfrak{X}}_K^{\mathrm{an}}$. We will work around this later.

Logarithmic extensions

We say $\mathcal{E} \in \mathbf{FIsoc}^{\dagger}(X)$ is **tame** if for every $z \in Z$, after making a base extension to ensure that $z \in Z(k)$, the pullback of $\mathcal{E}$ to $\mathbf{FIsoc}^{\dagger}(s)$ defined by z is tame (i.e., its local monodromy representation is tame).

Theorem

$\mathcal{E}$ is tame iff it extends to a vector bundle with logarithmic connection on $(\overline{\mathfrak{X}}_K^{\mathrm{an}}, \mathfrak{Z}_K^{\mathrm{an}})$ with exponents in $\mathbb{Z}_{(p)}$. In this case, there is a unique such extension with exponents in $\mathbb{Z}_{(p)} \cap [0,1)$.

Beware that we cannot include the Frobenius structure in this statement because in general there is no Frobenius lift on all of $\overline{\mathfrak{X}}_K^{\mathrm{an}}$. We will work around this later.

Logarithmic extensions

We say $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$ is **tame** if for every $z \in Z$, after making a base extension to ensure that $z \in Z(k)$, the pullback of $\mathcal{E}$ to $\mathbf{FIsoc}^\dagger(s)$ defined by z is tame (i.e., its local monodromy representation is tame).

Theorem

$\mathcal{E}$ is tame iff it extends to a vector bundle with logarithmic connection on $(\overline{\mathfrak{X}}_K^{\mathrm{an}}, \mathfrak{Z}_K^{\mathrm{an}})$ with exponents in $\mathbb{Z}_{(p)}$. In this case, there is a unique such extension with exponents in $\mathbb{Z}_{(p)} \cap [0,1)$.

Beware that we cannot include the Frobenius structure in this statement because in general there is no Frobenius lift on all of $\overline{\mathfrak{X}}_K^{\mathrm{an}}$. We will work around this later.

GAGA+GAGA

Theorem

If $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$ is tame, then any logarithmic extension on $(\overline{\mathfrak{X}}_K^{\mathrm{an}}, \mathfrak{Z}_K^{\mathrm{an}})$ with exponents in $\mathbb{Z}_{(p)}$ is the pullback of a vector bundle with logarithmic connection on the log scheme $(\overline{\mathfrak{X}}_K, \mathfrak{Z}_K)$.

This follows from the previous statement using the analogue of Serre's GAGA[9] theorem with $\mathbb{C}$ replaced by K.

On the other hand, since K is a field of characteristic 0 of cardinality $2^{\aleph_0}$, it admits an algebraic (but not topological!) embedding into $\mathbb{C}$. We can thus choose such an embedding, pull back from $\overline{\mathfrak{X}}_K$ to $\overline{\mathfrak{X}}_{\mathbb{C}}$, then apply results of complex analytic geometry via standard GAGA.

[9] Acronym of Serre's paper "Géométrie Algébrique et Géométrie Analytique".

GAGA+GAGA

Theorem

If $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$ is tame, then any logarithmic extension on $(\overline{\mathfrak{X}}_K^{\mathrm{an}}, \mathfrak{Z}_K^{\mathrm{an}})$ with exponents in $\mathbb{Z}_{(p)}$ is the pullback of a vector bundle with logarithmic connection on the log scheme $(\overline{\mathfrak{X}}_K, \mathfrak{Z}_K)$.

This follows from the previous statement using the analogue of Serre's GAGA[9] theorem with $\mathbb{C}$ replaced by K.

On the other hand, since K is a field of characteristic 0 of cardinality $2^{\aleph_0}$, it admits an algebraic (but not topological!) embedding into $\mathbb{C}$. We can thus choose such an embedding, pull back from $\overline{\mathfrak{X}}_K$ to $\overline{\mathfrak{X}}_{\mathbb{C}}$, then apply results of complex analytic geometry via standard GAGA.

[9] Acronym of Serre's paper "Géométrie Algébrique et Géométrie Analytique".

GAGA+GAGA

Theorem

If $\mathcal{E} \in \mathbf{FIsoc}^{\dagger}(X)$ is tame, then any logarithmic extension on $(\overline{\mathfrak{X}}_K^{\mathrm{an}}, \mathfrak{Z}_K^{\mathrm{an}})$ with exponents in $\mathbb{Z}_{(p)}$ is the pullback of a vector bundle with logarithmic connection on the log scheme $(\overline{\mathfrak{X}}_K, \mathfrak{Z}_K)$.

This follows from the previous statement using the analogue of Serre's GAGA[9] theorem with $\mathbb{C}$ replaced by K.

On the other hand, since K is a field of characteristic 0 of cardinality $2^{\aleph_0}$, it admits an algebraic (but not topological!) embedding into $\mathbb{C}$. We can thus choose such an embedding, pull back from $\overline{\mathfrak{X}}_K$ to $\overline{\mathfrak{X}}_{\mathbb{C}}$, then apply results of complex analytic geometry via standard GAGA.

[9] Acronym of Serre's paper "Géométrie Algébrique et Géométrie Analytique".

An example of GAGA+GAGA

Theorem

Suppose that $\mathcal{E} \in \mathbf{FIsoc}^{\dagger}(X)$ is tame with nilpotent residues, and let E be its logarithmic extension to $(\overline{\mathfrak{X}}_K, \mathfrak{Z}_K)$ with nilpotent residues. Then the first Chern class of E is zero; in particular, $\deg(E) = 0$.

Proof.

It is equivalent to check the claim after replacing $\mathcal{E}$ and E with their top exterior powers. Using GAGA+GAGA, we reduce to the statement that a line bundle on a compact Riemann surface admitting a connection (with no logarithmic singularities) has degree 0. This case is due to Atiyah. □

An example of GAGA+GAGA

Theorem

Suppose that $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$ *is tame with nilpotent residues, and let* E *be its logarithmic extension to* $(\overline{\mathfrak{X}}_K, \mathfrak{Z}_K)$ *with nilpotent residues. Then the first Chern class of* E *is zero; in particular,* $\deg(E) = 0$*.*

Proof.

It is equivalent to check the claim after replacing $\mathcal{E}$ and E with their top exterior powers. Using GAGA+GAGA, we reduce to the statement that a line bundle on a compact Riemann surface admitting a connection (with no logarithmic singularities) has degree 0. This case is due to Atiyah. □

Contents

Newton polygons and the jumping locus

For $r, s \in \mathbb{Z}$ with $r > 0$ and $\gcd(r, s) = 1$, the formula

$$F(\mathbf{e}_1) = \mathbf{e}_2, \quad \ldots, \quad F(\mathbf{e}_{r-1}) = \mathbf{e}_r, \quad F(\mathbf{e}_r) = p^s \mathbf{e}_1.$$

defines an object $\mathcal{E}_{s,r} \in \mathbf{FIsoc}^\dagger(\operatorname{Spec} k)$.

Theorem (Dieudonné–Manin)

For $k = \overline{k}$, every object $\mathcal{E} \in \mathbf{FIsoc}^\dagger(\operatorname{Spec} k)$ decomposes as a direct sum of various objects of the form $\mathcal{E}_{s,r}$. This decomposition is not unique in general, but the associated isotypical decomposition is unique.

We associate to $\mathcal{E}$ the **Newton polygon** with the slope $\frac{s}{r}$ with multiplicity equal to the rank of the isotypical summand corresponding to $\mathcal{E}_{s,r}$.

This behaves like the Newton polygon of a linear operator on a vector space over $\mathbb{Q}_p$. In particular, it behaves well with respect to tensor/symmetric/exterior powers and duals.

Newton polygons and the jumping locus

For $r, s \in \mathbb{Z}$ with $r > 0$ and $\gcd(r, s) = 1$, the formula

$$F(\mathbf{e}_1) = \mathbf{e}_2, \quad \dots, \quad F(\mathbf{e}_{r-1}) = \mathbf{e}_r, \quad F(\mathbf{e}_r) = p^s \mathbf{e}_1.$$

defines an object $\mathcal{E}_{s,r} \in \mathbf{FIsoc}^\dagger(\operatorname{Spec} k)$.

Theorem (Dieudonné–Manin)

For $k = \overline{k}$, every object $\mathcal{E} \in \mathbf{FIsoc}^\dagger(\operatorname{Spec} k)$ decomposes as a direct sum of various objects of the form $\mathcal{E}_{s,r}$. This decomposition is not unique in general, but the associated isotypical decomposition is unique.

We associate to $\mathcal{E}$ the **Newton polygon** with the slope $\frac{s}{r}$ with multiplicity equal to the rank of the isotypical summand corresponding to $\mathcal{E}_{s,r}$.

This behaves like the Newton polygon of a linear operator on a vector space over $\mathbb{Q}_p$. In particular, it behaves well with respect to tensor/symmetric/exterior powers and duals.

Newton polygons and the jumping locus

For $r, s \in \mathbb{Z}$ with $r > 0$ and $\gcd(r, s) = 1$, the formula

$$F(\mathbf{e}_1) = \mathbf{e}_2, \quad \dots, \quad F(\mathbf{e}_{r-1}) = \mathbf{e}_r, \quad F(\mathbf{e}_r) = p^s \mathbf{e}_1.$$

defines an object $\mathcal{E}_{s,r} \in \mathbf{FIsoc}^\dagger(\operatorname{Spec} k)$.

Theorem (Dieudonné–Manin)

For $k = \overline{k}$, every object $\mathcal{E} \in \mathbf{FIsoc}^\dagger(\operatorname{Spec} k)$ decomposes as a direct sum of various objects of the form $\mathcal{E}_{s,r}$. This decomposition is not unique in general, but the associated isotypical decomposition is unique.

We associate to $\mathcal{E}$ the **Newton polygon** with the slope $\frac{s}{r}$ with multiplicity equal to the rank of the isotypical summand corresponding to $\mathcal{E}_{s,r}$.

This behaves like the Newton polygon of a linear operator on a vector space over $\mathbb{Q}_p$. In particular, it behaves well with respect to tensor/symmetric/exterior powers and duals.

Newton polygons and the jumping locus

For $r, s \in \mathbb{Z}$ with $r > 0$ and $\gcd(r, s) = 1$, the formula

$$F(\mathbf{e}_1) = \mathbf{e}_2, \quad \dots, \quad F(\mathbf{e}_{r-1}) = \mathbf{e}_r, \quad F(\mathbf{e}_r) = p^s \mathbf{e}_1.$$

defines an object $\mathcal{E}_{s,r} \in \mathbf{FIsoc}^\dagger(\operatorname{Spec} k)$.

Theorem (Dieudonné–Manin)

For $k = \overline{k}$, every object $\mathcal{E} \in \mathbf{FIsoc}^\dagger(\operatorname{Spec} k)$ decomposes as a direct sum of various objects of the form $\mathcal{E}_{s,r}$. This decomposition is not unique in general, but the associated isotypical decomposition is unique.

We associate to $\mathcal{E}$ the **Newton polygon** with the slope $\frac{s}{r}$ with multiplicity equal to the rank of the isotypical summand corresponding to $\mathcal{E}_{s,r}$.

This behaves like the Newton polygon of a linear operator on a vector space over $\mathbb{Q}_p$. In particular, it behaves well with respect to tensor/symmetric/exterior powers and duals.

Semicontinuity for Newton polygons

For $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$, for any point $x \in X$ (including the generic point η), define the Newton polygon of $\mathcal{E}$ at x by pullback to a geometric point over x (it does not matter which one).

Theorem (Grothendieck–Katz, 1960s)

For $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$, the Newton polygon of $\mathcal{E}$ at x lies on or above the Newton polygon at η. Moreover, the endpoints always stay the same, and equality holds for x in some open dense subset of X.

Define the **jumping locus** of $\mathcal{E}$ as the set of $x \in X$ at which the Newton polygon does not agree with η.

Semicontinuity for Newton polygons

For $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$, for any point $x \in X$ (including the generic point η), define the Newton polygon of $\mathcal{E}$ at x by pullback to a geometric point over x (it does not matter which one).

Theorem (Grothendieck–Katz, 1960s)

For $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$, the Newton polygon of $\mathcal{E}$ at x lies on or above the Newton polygon at η. Moreover, the endpoints always stay the same, and equality holds for x in some open dense subset of X.

Define the **jumping locus** of $\mathcal{E}$ as the set of $x \in X$ at which the Newton polygon does not agree with η.

Semicontinuity for Newton polygons

For $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$, for any point $x \in X$ (including the generic point η), define the Newton polygon of $\mathcal{E}$ at x by pullback to a geometric point over x (it does not matter which one).

Theorem (Grothendieck–Katz, 1960s)

For $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$, the Newton polygon of $\mathcal{E}$ at x lies on or above the Newton polygon at η. Moreover, the endpoints always stay the same, and equality holds for x in some open dense subset of X.

Define the **jumping locus** of $\mathcal{E}$ as the set of $x \in X$ at which the Newton polygon does not agree with η.

Bounding the jumping locus

Theorem (Tsuzuki, K)

For $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$ tame, the length of the jumping locus can be bounded in terms of p, g, m, r.

For k finite, this can be proved using L-functions. For general k, it will follow from uniformity for crystalline lattices (see below).

In many cases, one can compute the **exact** length of the jumping locus using "transversality of Frobenius" (e.g., on modular curves). Is there a general result of this form?

Bounding the jumping locus

Theorem (Tsuzuki, K)

For $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$ tame, the length of the jumping locus can be bounded in terms of p, g, m, r.

For k finite, this can be proved using L-functions. For general k, it will follow from uniformity for crystalline lattices (see below).

In many cases, one can compute the **exact** length of the jumping locus using "transversality of Frobenius" (e.g., on modular curves). Is there a general result of this form?

Bounding the jumping locus

Theorem (Tsuzuki, K)

For $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$ tame, the length of the jumping locus can be bounded in terms of p, g, m, r.

For k finite, this can be proved using L-functions. For general k, it will follow from uniformity for crystalline lattices (see below).

In many cases, one can compute the **exact** length of the jumping locus using "transversality of Frobenius" (e.g., on modular curves). Is there a general result of this form?

Contents

Crystalline lattices

For $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$ tame with nilpotent residues, let E_K be the logarithmic extension to $(\overline{\mathfrak{X}}_K, \mathfrak{Z}_K)$ with nilpotent residues.

A **lattice** in $\mathcal{E}$ is an extension of E_K to a vector bundle[10] E with logarithmic connection on $(\overline{\mathfrak{X}}, \mathfrak{Z})$.

Let η_X be the generic point of X. A lattice E is **crystalline** if its pullback to the completed localization of $\overline{\mathfrak{X}}$ at η_X is stable under the action on $\mathcal{E}$ of any Frobenius structure.

Theorem

$\mathcal{E}$ admits a crystalline lattice if and only if its Newton polygon at η_X has all nonnegative slopes. (The same is then true everywhere on X.)

[10] Since $\overline{\mathfrak{X}}$ is regular of dimension 2, any **reflexive** extension is already a vector bundle.

Crystalline lattices

For $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$ tame with nilpotent residues, let E_K be the logarithmic extension to $(\overline{\mathfrak{X}}_K, \mathfrak{Z}_K)$ with nilpotent residues.

A **lattice** in $\mathcal{E}$ is an extension of E_K to a vector bundle[10] E with logarithmic connection on $(\overline{\mathfrak{X}}, \mathfrak{Z})$.

Let η_X be the generic point of X. A lattice E is **crystalline** if its pullback to the completed localization of $\overline{\mathfrak{X}}$ at η_X is stable under the action on $\mathcal{E}$ of any Frobenius structure.

Theorem

$\mathcal{E}$ admits a crystalline lattice if and only if its Newton polygon at η_X has all nonnegative slopes. (The same is then true everywhere on X.)

[10] Since $\overline{\mathfrak{X}}$ is regular of dimension 2, any **reflexive** extension is already a vector bundle.

Crystalline lattices

For $\mathcal{E} \in \mathbf{FIsoc}^{\dagger}(X)$ tame with nilpotent residues, let E_K be the logarithmic extension to $(\overline{\mathfrak{X}}_K, \mathfrak{Z}_K)$ with nilpotent residues.

A **lattice** in $\mathcal{E}$ is an extension of E_K to a vector bundle[10] E with logarithmic connection on $(\overline{\mathfrak{X}}, \mathfrak{Z})$.

Let η_X be the generic point of X. A lattice E is **crystalline** if its pullback to the completed localization of $\overline{\mathfrak{X}}$ at η_X is stable under the action on $\mathcal{E}$ of any Frobenius structure.

Theorem

$\mathcal{E}$ admits a crystalline lattice if and only if its Newton polygon at η_X has all nonnegative slopes. (The same is then true everywhere on X.)

[10] Since $\overline{\mathfrak{X}}$ is regular of dimension 2, any **reflexive** extension is already a vector bundle.

Crystalline lattices

For $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$ tame with nilpotent residues, let E_K be the logarithmic extension to $(\overline{\mathfrak{X}}_K, \mathfrak{Z}_K)$ with nilpotent residues.

A **lattice** in $\mathcal{E}$ is an extension of E_K to a vector bundle[10] E with logarithmic connection on $(\overline{\mathfrak{X}}, \mathfrak{Z})$.

Let η_X be the generic point of X. A lattice E is **crystalline** if its pullback to the completed localization of $\overline{\mathfrak{X}}$ at η_X is stable under the action on $\mathcal{E}$ of any Frobenius structure.

Theorem

$\mathcal{E}$ admits a crystalline lattice if and only if its Newton polygon at η_X has all nonnegative slopes. (The same is then true everywhere on X.)

[10] Since $\overline{\mathfrak{X}}$ is regular of dimension 2, any **reflexive** extension is already a vector bundle.

Crystalline lattices and the global action of Frobenius

In the definition of a crystalline lattice E, there is no way to say directly that "E is preserved by Frobenius structures" because the latter are not defined everywhere on $\overline{X}_K^{\mathrm{an}}$.

However, we do get a well-defined Frobenius action on the pullback E_k of E to $\overline{X}$. This is not an isomorphism: its generic rank is the multiplicity of 0 in the Newton polygon at η_X.

By the same token, the rank of the Frobenius action at any $x \in X$ is the multiplicity of 0 in the Newton polygon at x. This means that we can control the length of the locus at which this multiplicity drops by bounding the degree of the image of Frobenius on E_k.

Hence using this construction for $\mathcal{E}$ and suitable twists of its exterior powers, we can detect the jumping locus of $\mathcal{E}$.

Crystalline lattices and the global action of Frobenius

In the definition of a crystalline lattice E, there is no way to say directly that "E is preserved by Frobenius structures" because the latter are not defined everywhere on $\overline{X}_K^{\mathrm{an}}$.

However, we do get a well-defined Frobenius action on the pullback E_k of E to $\overline{X}$. This is not an isomorphism: its generic rank is the multiplicity of 0 in the Newton polygon at η_X.

By the same token, the rank of the Frobenius action at any $x \in X$ is the multiplicity of 0 in the Newton polygon at x. This means that we can control the length of the locus at which this multiplicity drops by bounding the degree of the image of Frobenius on E_k.

Hence using this construction for $\mathcal{E}$ and suitable twists of its exterior powers, we can detect the jumping locus of $\mathcal{E}$.

Crystalline lattices and the global action of Frobenius

In the definition of a crystalline lattice E, there is no way to say directly that "E is preserved by Frobenius structures" because the latter are not defined everywhere on $\overline{X}_K^{\mathrm{an}}$.

However, we do get a well-defined Frobenius action on the pullback E_k of E to $\overline{X}$. This is not an isomorphism: its generic rank is the multiplicity of 0 in the Newton polygon at η_X.

By the same token, the rank of the Frobenius action at any $x \in X$ is the multiplicity of 0 in the Newton polygon at x. This means that we can control the length of the locus at which this multiplicity drops by bounding the degree of the image of Frobenius on E_k.

Hence using this construction for $\mathcal{E}$ and suitable twists of its exterior powers, we can detect the jumping locus of $\mathcal{E}$.

Crystalline lattices and the global action of Frobenius

In the definition of a crystalline lattice E, there is no way to say directly that "E is preserved by Frobenius structures" because the latter are not defined everywhere on $\overline{X}_K^{\mathrm{an}}$.

However, we do get a well-defined Frobenius action on the pullback E_k of E to $\overline{X}$. This is not an isomorphism: its generic rank is the multiplicity of 0 in the Newton polygon at η_X.

By the same token, the rank of the Frobenius action at any $x \in X$ is the multiplicity of 0 in the Newton polygon at x. This means that we can control the length of the locus at which this multiplicity drops by bounding the degree of the image of Frobenius on E_k.

Hence using this construction for $\mathcal{E}$ and suitable twists of its exterior powers, we can detect the jumping locus of $\mathcal{E}$.

Slopes of vector bundles

Let C be a curve over a field L of any characteristic. Let F be a vector bundle on C.

- The **determinant** $\det(F) = \wedge^{\mathrm{rank}(F)} F$.
- The **degree** $\deg(F) := \deg(\det(F))$ where degree of a line bundle means the degree of a nonzero rational section.
- For $F \neq 0$, the **slope** $\mu(F) := \frac{\deg(F)}{\mathrm{rank}(F)}$. If $H^0(C, F) \neq 0$ then $\mu(F) \geq 0$; the converse is false, but if $\mu(F) > 2g - 2$ then $H^0(C, F) \neq 0$ (Riemann-Roch).
- For $F \neq 0$, F is **stable** (resp. **semistable**) if there is no nonzero proper subbundle G of F with $\mu(G) \geq \mu(F)$ (resp. $\mu(G) > \mu(F)$).

Slopes of vector bundles

Let C be a curve over a field L of any characteristic. Let F be a vector bundle on C.

- The **determinant** $\det(F) = \wedge^{\mathrm{rank}(F)} F$.
- The **degree** $\deg(F) := \deg(\det(F))$ where degree of a line bundle means the degree of a nonzero rational section.
- For $F \neq 0$, the **slope** $\mu(F) := \frac{\deg(F)}{\mathrm{rank}(F)}$. If $H^0(C, F) \neq 0$ then $\mu(F) \geq 0$; the converse is false, but if $\mu(F) > 2g - 2$ then $H^0(C, F) \neq 0$ (Riemann-Roch).
- For $F \neq 0$, F is **stable** (resp. **semistable**) if there is no nonzero proper subbundle G of F with $\mu(G) \geq \mu(F)$ (resp. $\mu(G) > \mu(F)$).

Slopes of vector bundles

Let C be a curve over a field L of any characteristic. Let F be a vector bundle on C.

- The **determinant** $\det(F) = \wedge^{\mathrm{rank}(F)} F$.
- The **degree** $\deg(F) := \deg(\det(F))$ where degree of a line bundle means the degree of a nonzero rational section.
- For $F \neq 0$, the **slope** $\mu(F) := \frac{\deg(F)}{\mathrm{rank}(F)}$. If $H^0(C, F) \neq 0$ then $\mu(F) \geq 0$; the converse is false, but if $\mu(F) > 2g - 2$ then $H^0(C, F) \neq 0$ (Riemann-Roch).
- For $F \neq 0$, F is **stable** (resp. **semistable**) if there is no nonzero proper subbundle G of F with $\mu(G) \geq \mu(F)$ (resp. $\mu(G) > \mu(F)$).

Slopes of vector bundles

Let C be a curve over a field L of any characteristic. Let F be a vector bundle on C.

- The **determinant** $\det(F) = \wedge^{\mathrm{rank}(F)} F$.
- The **degree** $\deg(F) := \deg(\det(F))$ where degree of a line bundle means the degree of a nonzero rational section.
- For $F \neq 0$, the **slope** $\mu(F) := \frac{\deg(F)}{\mathrm{rank}(F)}$. If $H^0(C, F) \neq 0$ then $\mu(F) \geq 0$; the converse is false, but if $\mu(F) > 2g - 2$ then $H^0(C, F) \neq 0$ (Riemann-Roch).
- For $F \neq 0$, F is **stable** (resp. **semistable**) if there is no nonzero proper subbundle G of F with $\mu(G) \geq \mu(F)$ (resp. $\mu(G) > \mu(F)$).

Slopes of vector bundles

Let C be a curve over a field L of any characteristic. Let F be a vector bundle on C.

- The **determinant** $\det(F) = \wedge^{\mathrm{rank}(F)} F$.
- The **degree** $\deg(F) := \deg(\det(F))$ where degree of a line bundle means the degree of a nonzero rational section.
- For $F \neq 0$, the **slope** $\mu(F) := \frac{\deg(F)}{\mathrm{rank}(F)}$. If $H^0(C, F) \neq 0$ then $\mu(F) \geq 0$; the converse is false, but if $\mu(F) > 2g - 2$ then $H^0(C, F) \neq 0$ (Riemann-Roch).
- For $F \neq 0$, F is **stable** (resp. **semistable**) if there is no nonzero proper subbundle G of F with $\mu(G) \geq \mu(F)$ (resp. $\mu(G) > \mu(F)$).

Harder–Narasimhan polygons

For C, L, F as above, there exists a unique filtration

$$F = F_0 \supset \cdots \supset F_l = 0$$

such that:

(a) each successive quotient F_i/F_{i+1} is semistable;

(b) for $\mu_i := \mu(F_i/F_{i+1})$, we have $\mu_1 > \cdots > \mu_l$.

This is the **HN (Harder–Narasimhan) filtration** of F. The **HN polygon** of F is the Newton polygon with slope μ_i of multiplicity $\mathrm{rank}(F_i/F_{i+1})$.

The HN polygon is semicontinuous under specialization. In particular, for E a crystalline lattice, bounding the HN polygon of E_k also bounds the HN polygon of E_K, but not conversely.

Harder–Narasimhan polygons

For C, L, F as above, there exists a unique filtration

$$F = F_0 \supset \cdots \supset F_l = 0$$

such that:

(a) each successive quotient F_i/F_{i+1} is semistable;

(b) for $\mu_i := \mu(F_i/F_{i+1})$, we have $\mu_1 > \cdots > \mu_l$.

This is the **HN (Harder–Narasimhan) filtration** of F. The **HN polygon** of F is the Newton polygon with slope μ_i of multiplicity $\mathrm{rank}(F_i/F_{i+1})$.

The HN polygon is semicontinuous under specialization. In particular, for E a crystalline lattice, bounding the HN polygon of E_k also bounds the HN polygon of E_K, but not conversely.

Harder–Narasimhan polygons

For C, L, F as above, there exists a unique filtration

$$F = F_0 \supset \cdots \supset F_l = 0$$

such that:

(a) each successive quotient F_i/F_{i+1} is semistable;

(b) for $\mu_i := \mu(F_i/F_{i+1})$, we have $\mu_1 > \cdots > \mu_l$.

This is the **HN (Harder–Narasimhan) filtration** of F. The **HN polygon** of F is the Newton polygon with slope μ_i of multiplicity $\mathrm{rank}(F_i/F_{i+1})$.

The HN polygon is semicontinuous under specialization. In particular, for E a crystalline lattice, bounding the HN polygon of E_k also bounds the HN polygon of E_K, but not conversely.

Harder–Narasimhan polygons

For C, L, F as above, there exists a unique filtration

$$F = F_0 \supset \cdots \supset F_l = 0$$

such that:

(a) each successive quotient F_i/F_{i+1} is semistable;

(b) for $\mu_i := \mu(F_i/F_{i+1})$, we have $\mu_1 > \cdots > \mu_l$.

This is the **HN (Harder–Narasimhan) filtration** of F. The **HN polygon** of F is the Newton polygon with slope μ_i of multiplicity $\mathrm{rank}(F_i/F_{i+1})$.

The HN polygon is semicontinuous under specialization. In particular, for E a crystalline lattice, bounding the HN polygon of E_k also bounds the HN polygon of E_K, but not conversely.

Harder–Narasimhan polygons

For C, L, F as above, there exists a unique filtration

$$F = F_0 \supset \cdots \supset F_l = 0$$

such that:

(a) each successive quotient F_i/F_{i+1} is semistable;

(b) for $\mu_i := \mu(F_i/F_{i+1})$, we have $\mu_1 > \cdots > \mu_l$.

This is the **HN (Harder–Narasimhan) filtration** of F. The **HN polygon** of F is the Newton polygon with slope μ_i of multiplicity $\mathrm{rank}(F_i/F_{i+1})$.

The HN polygon is semicontinuous under specialization. In particular, for E a crystalline lattice, bounding the HN polygon of E_k also bounds the HN polygon of E_K, but not conversely.

Uniformity for crystalline lattices

Theorem

For $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$ tame with nilpotent residues of rank r with nonnegative Newton slopes, there exists a crystalline lattice E such that the HN polygon of E_k is bounded (above and below) in terms of p, g, m, r.

The key point: if $\mathcal{E}$ is irreducible, then the gaps between consecutive HN slopes of E_k are bounded. To wit, a large gap implies an exact sequence

$$0 \to E_{k,1} \to E_k \to E_{k,2} \to 0$$

in which $E_{k,1}$ is forced to be stable under the Frobenius and connection. We then get a new lattice E' with an exact sequence

$$0 \to E_{k,2} \to E'_k \to E_{k,1} \to 0.$$

These steps form a walk through a **bounded** subset of the Bruhat–Tits building for $\mathrm{GL}_r(\mathbb{Q}_p)$, which eventually terminates at a good lattice.

Uniformity for crystalline lattices

Theorem

For $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$ tame with nilpotent residues of rank r with nonnegative Newton slopes, there exists a crystalline lattice E such that the HN polygon of E_k is bounded (above and below) in terms of p, g, m, r.

The key point: if $\mathcal{E}$ is irreducible, then the gaps between consecutive HN slopes of E_k are bounded. To wit, a large gap implies an exact sequence

$$0 \to E_{k,1} \to E_k \to E_{k,2} \to 0$$

in which $E_{k,1}$ is forced to be stable under the Frobenius and connection. We then get a new lattice E' with an exact sequence

$$0 \to E_{k,2} \to E'_k \to E_{k,1} \to 0.$$

These steps form a walk through a **bounded** subset of the Bruhat–Tits building for $\mathrm{GL}_r(\mathbb{Q}_p)$, which eventually terminates at a good lattice.

Uniformity for crystalline lattices

Theorem

For $\mathcal{E} \in \mathbf{FIsoc}^\dagger(X)$ tame with nilpotent residues of rank r with nonnegative Newton slopes, there exists a crystalline lattice E such that the HN polygon of E_k is bounded (above and below) in terms of p, g, m, r.

The key point: if $\mathcal{E}$ is irreducible, then the gaps between consecutive HN slopes of E_k are bounded. To wit, a large gap implies an exact sequence

$$0 \to E_{k,1} \to E_k \to E_{k,2} \to 0$$

in which $E_{k,1}$ is forced to be stable under the Frobenius and connection. We then get a new lattice E' with an exact sequence

$$0 \to E_{k,2} \to E'_k \to E_{k,1} \to 0.$$

These steps form a walk through a **bounded** subset of the Bruhat–Tits building for $\mathrm{GL}_r(\mathbb{Q}_p)$, which eventually terminates at a good lattice.

Contents

A remark about coefficient fields

From now on, assume k is finite.

The category $\mathbf{FIsoc}^\dagger(X)$ is a $\mathbb{Q}_p$-linear tensor category. For any finite extension L of $\mathbb{Q}_p$, we may form the category $\mathbf{FIsoc}^\dagger(X) \otimes L$ consisting of objects of $\mathbf{FIsoc}^\dagger(X)$ equipped with a $\mathbb{Q}_p$-linear L-action. By taking a 2-colimit over L, we obtain the category $\mathbf{FIsoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$.

This is the p-adic analogue of the category of étale Weil $\overline{\mathbb{Q}}_\ell$-sheaves on X for some prime $\ell \neq p$. Forgetting[11] Frobenius actions gives an analogue of the category of étale lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $X_{\overline{k}}$.

In particular, $\mathbf{FIsoc}^\dagger(\operatorname{Spec} k) \otimes \overline{\mathbb{Q}}_p$ is equivalent to the category of finite-dimensional $\overline{\mathbb{Q}}_p$-vector spaces equipped with a $\overline{\mathbb{Q}}_p$-linear automorphism.

[11] This requires some care, as an isocrystal without Frobenius structure is not just a vector bundle with connection on a strict neighborhood; there is an extra convergence condition on the Taylor isomorphism.

A remark about coefficient fields

From now on, assume k is finite.

The category $\mathbf{FIsoc}^\dagger(X)$ is a $\mathbb{Q}_p$-linear tensor category. For any finite extension L of $\mathbb{Q}_p$, we may form the category $\mathbf{FIsoc}^\dagger(X) \otimes L$ consisting of objects of $\mathbf{FIsoc}^\dagger(X)$ equipped with a $\mathbb{Q}_p$-linear L-action. By taking a 2-colimit over L, we obtain the category $\mathbf{FIsoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$.

This is the p-adic analogue of the category of étale Weil $\overline{\mathbb{Q}}_\ell$-sheaves on X for some prime $\ell \neq p$. Forgetting[11] Frobenius actions gives an analogue of the category of étale lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $X_{\overline{k}}$.

In particular, $\mathbf{FIsoc}^\dagger(\operatorname{Spec} k) \otimes \overline{\mathbb{Q}}_p$ is equivalent to the category of finite-dimensional $\overline{\mathbb{Q}}_p$-vector spaces equipped with a $\overline{\mathbb{Q}}_p$-linear automorphism.

[11] This requires some care, as an isocrystal without Frobenius structure is not just a vector bundle with connection on a strict neighborhood; there is an extra convergence condition on the Taylor isomorphism.

A remark about coefficient fields

From now on, assume k is finite.

The category $\mathbf{FIsoc}^\dagger(X)$ is a $\mathbb{Q}_p$-linear tensor category. For any finite extension L of $\mathbb{Q}_p$, we may form the category $\mathbf{FIsoc}^\dagger(X) \otimes L$ consisting of objects of $\mathbf{FIsoc}^\dagger(X)$ equipped with a $\mathbb{Q}_p$-linear L-action. By taking a 2-colimit over L, we obtain the category $\mathbf{FIsoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$.

This is the p-adic analogue of the category of étale Weil $\overline{\mathbb{Q}}_\ell$-sheaves on X for some prime $\ell \neq p$. Forgetting[11] Frobenius actions gives an analogue of the category of étale lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $X_{\overline{k}}$.

In particular, $\mathbf{FIsoc}^\dagger(\operatorname{Spec} k) \otimes \overline{\mathbb{Q}}_p$ is equivalent to the category of finite-dimensional $\overline{\mathbb{Q}}_p$-vector spaces equipped with a $\overline{\mathbb{Q}}_p$-linear automorphism.

[11] This requires some care, as an isocrystal without Frobenius structure is not just a vector bundle with connection on a strict neighborhood; there is an extra convergence condition on the Taylor isomorphism.

A remark about coefficient fields

From now on, assume k is finite.

The category $\mathbf{FIsoc}^\dagger(X)$ is a $\mathbb{Q}_p$-linear tensor category. For any finite extension L of $\mathbb{Q}_p$, we may form the category $\mathbf{FIsoc}^\dagger(X) \otimes L$ consisting of objects of $\mathbf{FIsoc}^\dagger(X)$ equipped with a $\mathbb{Q}_p$-linear L-action. By taking a 2-colimit over L, we obtain the category $\mathbf{FIsoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p$.

This is the p-adic analogue of the category of étale Weil $\overline{\mathbb{Q}}_\ell$-sheaves on X for some prime $\ell \neq p$. Forgetting[11] Frobenius actions gives an analogue of the category of étale lisse $\overline{\mathbb{Q}}_\ell$-sheaves on $X_{\overline{k}}$.

In particular, $\mathbf{FIsoc}^\dagger(\operatorname{Spec} k) \otimes \overline{\mathbb{Q}}_p$ is equivalent to the category of finite-dimensional $\overline{\mathbb{Q}}_p$-vector spaces equipped with a $\overline{\mathbb{Q}}_p$-linear automorphism.

[11] This requires some care, as an isocrystal without Frobenius structure is not just a vector bundle with connection on a strict neighborhood; there is an extra convergence condition on the Taylor isomorphism.

The theorem on crystalline companions

Theorem (K, 2023)

Let Y be a smooth k-scheme. Fix on $\overline{\mathbb{Q}}$ a place v_1 above $\ell \neq p$ and a place v_2 above p. Let $\mathcal{E}$ be an étale lisse $\overline{\mathbb{Q}}_\ell$-sheaf which is irreducible with finite determinant. Then there exists a unique $\mathcal{F} \in \mathbf{FIsoc}^\dagger(Y) \otimes \overline{\mathbb{Q}}_p$ such that at each $y \in Y$, the characteristic polynomials of Frob_y on $\mathcal{E}$ and $\mathcal{F}$ coincide in $\overline{\mathbb{Q}}[T]$ (using v_1 and v_2 to construct the embeddings $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$.

For $\dim(Y) = 1$ and $r = \mathrm{rank}(\mathcal{E})$, this follows from the Langlands correspondence for GL_r with ℓ-adic coefficients (Drinfeld, L. Lafforgue) and p-adic coefficients (T. Abe). The challenge here is to apply this result to all curves in Y, then use the result to obtain something coherent.

The analogous statement for v_1, v_2 away from p follows from work of Deligne and Drinfeld. This was extended to v_1 above p, v_2 away from p by Abe–Esnault and K.

The theorem on crystalline companions

Theorem (K, 2023)

Let Y be a smooth k-scheme. Fix on $\overline{\mathbb{Q}}$ a place v_1 above $\ell \neq p$ and a place v_2 above p. Let $\mathcal{E}$ be an étale lisse $\overline{\mathbb{Q}}_\ell$-sheaf which is irreducible with finite determinant. Then there exists a unique $\mathcal{F} \in \mathbf{FIsoc}^\dagger(Y) \otimes \overline{\mathbb{Q}}_p$ such that at each $y \in Y$, the characteristic polynomials of Frob_y on $\mathcal{E}$ and $\mathcal{F}$ coincide in $\overline{\mathbb{Q}}[T]$ (using v_1 and v_2 to construct the embeddings $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$.

For $\dim(Y) = 1$ and $r = \mathrm{rank}(\mathcal{E})$, this follows from the Langlands correspondence for GL_r with ℓ-adic coefficients (Drinfeld, L. Lafforgue) and p-adic coefficients (T. Abe). The challenge here is to apply this result to all curves in Y, then use the result to obtain something coherent.

The analogous statement for v_1, v_2 away from p follows from work of Deligne and Drinfeld. This was extended to v_1 above p, v_2 away from p by Abe–Esnault and K.

The theorem on crystalline companions

Theorem (K, 2023)

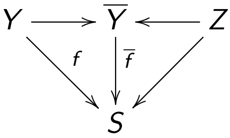
Let Y be a smooth k-scheme. Fix on $\overline{\mathbb{Q}}$ a place v_1 above $\ell \neq p$ and a place v_2 above p. Let $\mathcal{E}$ be an étale lisse $\overline{\mathbb{Q}}_\ell$-sheaf which is irreducible with finite determinant. Then there exists a unique $\mathcal{F} \in \mathbf{FIsoc}^\dagger(Y) \otimes \overline{\mathbb{Q}}_p$ such that at each $y \in Y$, the characteristic polynomials of Frob_y on $\mathcal{E}$ and $\mathcal{F}$ coincide in $\overline{\mathbb{Q}}[T]$ (using v_1 and v_2 to construct the embeddings $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell, \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$.

For $\dim(Y) = 1$ and $r = \operatorname{rank}(\mathcal{E})$, this follows from the Langlands correspondence for GL_r with ℓ-adic coefficients (Drinfeld, L. Lafforgue) and p-adic coefficients (T. Abe). The challenge here is to apply this result to all curves in Y, then use the result to obtain something coherent.

The analogous statement for v_1, v_2 away from p follows from work of Deligne and Drinfeld. This was extended to v_1 above p, v_2 away from p by Abe–Esnault and K.

A key geometric setup

Thanks to prior results, this can be checked after an alteration. So we may assume $\mathcal{E}$ is everywhere tame **and** that there is a diagram



which is an **elementary fibration** in the sense of Artin:

- S is smooth over k;
- $\overline{Y} \to S$ is a family of smooth projective curves;
- $Z \to \overline{Y}$ is a closed immersion with $Z \to S$ finite étale;
- $Y = \overline{Y} \setminus Z$.

We may also proceed by induction, so we may assume the existence of companions on both fibers and multisections of f.

A key geometric setup

Thanks to prior results, this can be checked after an alteration. So we may assume $\mathcal{E}$ is everywhere tame **and** that there is a diagram

$$\begin{array}{ccccc} Y & \longrightarrow & \overline{Y} & \longleftarrow & Z \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle \overline{f}} & \swarrow & \\ & & S & & \end{array}$$

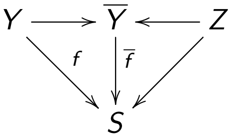
which is an **elementary fibration** in the sense of Artin:

- S is smooth over k;
- $\overline{Y} \to S$ is a family of smooth projective curves;
- $Z \to \overline{Y}$ is a closed immersion with $Z \to S$ finite étale;
- $Y = \overline{Y} \setminus Z$.

We may also proceed by induction, so we may assume the existence of companions on both fibers and multisections of f.

A key geometric setup

Thanks to prior results, this can be checked after an alteration. So we may assume $\mathcal{E}$ is everywhere tame **and** that there is a diagram



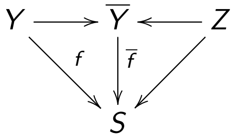
which is an **elementary fibration** in the sense of Artin:

- S is smooth over k;
- $\overline{Y} \to S$ is a family of smooth projective curves;
- $Z \to \overline{Y}$ is a closed immersion with $Z \to S$ finite étale;
- $Y = \overline{Y} \setminus Z$.

We may also proceed by induction, so we may assume the existence of companions on both fibers and multisections of f.

A key geometric setup

Thanks to prior results, this can be checked after an alteration. So we may assume $\mathcal{E}$ is everywhere tame **and** that there is a diagram



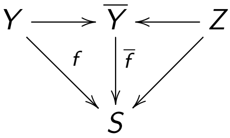
which is an **elementary fibration** in the sense of Artin:

- S is smooth over k;
- $\overline{Y} \to S$ is a family of smooth projective curves;
- $Z \to \overline{Y}$ is a closed immersion with $Z \to S$ finite étale;
- $Y = \overline{Y} \setminus Z$.

We may also proceed by induction, so we may assume the existence of companions on both fibers and multisections of f.

A key geometric setup

Thanks to prior results, this can be checked after an alteration. So we may assume $\mathcal{E}$ is everywhere tame **and** that there is a diagram



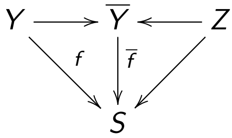
which is an **elementary fibration** in the sense of Artin:

- S is smooth over k;
- $\overline{Y} \to S$ is a family of smooth projective curves;
- $Z \to \overline{Y}$ is a closed immersion with $Z \to S$ finite étale;
- $Y = \overline{Y} \setminus Z$.

We may also proceed by induction, so we may assume the existence of companions on both fibers and multisections of f.

A key geometric setup

Thanks to prior results, this can be checked after an alteration. So we may assume $\mathcal{E}$ is everywhere tame **and** that there is a diagram



which is an **elementary fibration** in the sense of Artin:

- S is smooth over k;
- $\overline{Y} \to S$ is a family of smooth projective curves;
- $Z \to \overline{Y}$ is a closed immersion with $Z \to S$ finite étale;
- $Y = \overline{Y} \setminus Z$.

We may also proceed by induction, so we may assume the existence of companions on both fibers and multisections of f.

The role of uniformities

The method of Deligne to produce étale companions is to produce a coherent sequence of mod-ℓ^n truncations using a finiteness/compactness argument.

The analogous construction uses moduli stacks of (truncated) tame isocrystals. Uniformity for crystalline lattices implies that these are **finite type** over k.

Moreover, using uniformity for jumping loci on curves, we can show that the "jumping locus" of $\mathcal{E}$ behaves like we expect.

We then proceed by building the successive quotients of the slope filtration (via their associated p-adic local systems), then patch these together.

The role of uniformities

The method of Deligne to produce étale companions is to produce a coherent sequence of mod-ℓ^n truncations using a finiteness/compactness argument.

The analogous construction uses moduli stacks of (truncated) tame isocrystals. Uniformity for crystalline lattices implies that these are **finite type** over k.

Moreover, using uniformity for jumping loci on curves, we can show that the "jumping locus" of $\mathcal{E}$ behaves like we expect.

We then proceed by building the successive quotients of the slope filtration (via their associated p-adic local systems), then patch these together.

The role of uniformities

The method of Deligne to produce étale companions is to produce a coherent sequence of mod-ℓ^n truncations using a finiteness/compactness argument.

The analogous construction uses moduli stacks of (truncated) tame isocrystals. Uniformity for crystalline lattices implies that these are **finite type** over k.

Moreover, using uniformity for jumping loci on curves, we can show that the "jumping locus" of $\mathcal{E}$ behaves like we expect.

We then proceed by building the successive quotients of the slope filtration (via their associated p-adic local systems), then patch these together.

The role of uniformities

The method of Deligne to produce étale companions is to produce a coherent sequence of mod-ℓ^n truncations using a finiteness/compactness argument.

The analogous construction uses moduli stacks of (truncated) tame isocrystals. Uniformity for crystalline lattices implies that these are **finite type** over k.

Moreover, using uniformity for jumping loci on curves, we can show that the "jumping locus" of $\mathcal{E}$ behaves like we expect.

We then proceed by building the successive quotients of the slope filtration (via their associated p-adic local systems), then patch these together.