Slope filtrations and (ϕ, Γ)-modules in families

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unstable draft; version of February 25, 2010

These are the notes for a three-lecture minicourse given at the Institut Henri Poincaré in January 2010 as part of the Galois Trimester. The first lecture reviews the theory of slopes and slope filtrations for Frobenius actions (ϕ-modules) over the Robba ring, the link to p-adic Hodge theory via the work of Berger, and the analogue of Dieudonné-Manin classifications over the Robba ring. The second lecture introduces the notion of an arithmetic family of ϕ-modules, and describes our fairly limited knowledge about such objects, particularly the variation of slopes in a family. The few positive results we have are joint with Ruochuan Liu. The third lecture introduces the notion of a geometric family of ϕ-modules, gives a much more comprehensive treatment of variation of slopes than in the arithmetic case, and indicates an application to the theory of Rapoport-Zink period domains. This lecture represents work in progress, again joint with Ruochuan Liu; we plan to prepare a more detailed manuscript later.

Acknowledgments

Thanks to the organizers of the Galois Trimester for the invitation to deliver this minicourse, to Ruochuan Liu for feedback on the notes before the minicourse took place, to Fabrizio Andreatta for feedback on the lectures themselves, and to Jay Pottharst for subsequent feedback. Additional financial support was provided by NSF CAREER grant DMS-0545904, DARPA grant HR0011-09-1-0048, MIT (NEC Fund, Cecil and Ida Green Career Development Professorship), and the Institute for Advanced Study (NSF grant DMS-0635607, James D. Wolfensohn Fund).

1 Slope filtrations and (ϕ, Γ)-modules

In this lecture, we discuss the theory of slope filtrations for ϕ-modules over the Robba ring, and make the connection to Galois representations via the theory of (ϕ, Γ)-modules. We will work at a somewhat greater level of generality than might seem to be necessary at first; this generality will be needed in the later lectures.

1.1 ϕ-modules over a field

We start with a bit of "semilinear algebra" over a complete discretely valued field. This includes the Dieudonné-Manin classification theorem for rational Dieudonné modules over an algebraically closed field. This discussion is taken from [29, Chapter 14], for which see for additional references.

Hypothesis 1.1.1. Throughout § 1.1, let K be a field complete for a discrete valuation v, with residue field k of characteristic p. (We do not assume $p > 0$ except when specified.) Let $\mathfrak{o}_K$ be the valuation subring of K, and let $\mathfrak{m}_K$ be the maximal ideal of $\mathfrak{o}_K$. Let $\phi : K \to K$ be an endomorphism which is an isometry for v, so that ϕ induces an endomorphism $\overline{\phi} : k \to k$.

Definition 1.1.2. For V a K-vector space, let $\phi^* V = V \otimes_{K,\phi} K$ be the extension of scalars of V along ϕ, in which $\mathbf{v} \otimes \phi(r) = (r\mathbf{v}) \otimes 1$ for $r \in K, \mathbf{v} \in V$ and the scalar multiplication is defined by $r(\mathbf{v} \otimes s) = \mathbf{v} \otimes rs$ for $r, s \in K, \mathbf{v} \in V$. (Do not confuse $\phi^* V$ with the restriction of scalars $\phi_* V$.) Note that for W another K-vector space, given a K-linear map $A : \phi^* V \to W$, the map $B : V \to W$ given by $B(\mathbf{v}) = A(\mathbf{v} \otimes 1)$ is *ϕ-semilinear*, i.e., it is additive and satisfies $B(r\mathbf{v}) = \phi(r)B(\mathbf{v})$ for $r \in K, \mathbf{v} \in V$. Conversely, given a ϕ-semilinear map $B : V \to W$, the formula $A(\mathbf{v} \otimes r) = rB(\mathbf{v})$ defines a K-linear map $A : \phi^* V \to W$.

A *ϕ-module* over K is a finite-dimensional K-vector space V equipped with an isomorphism $\Phi : \phi^* V \to V$ of K-modules. By the previous paragraph, it is equivalent to equip V with a semilinear action of ϕ which carries any basis of V to another basis. In fact, it suffices to check this for a single basis, as may be seen as follows. Use one basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ to identify V with a space of column vectors, and define the matrix of action F of ϕ on $\mathbf{e}_1, \dots, \mathbf{e}_n$ by the formula

$$\phi(\mathbf{e}_j) = \sum_i F_{ij} \mathbf{e}_i.$$

Then define the change-of-basis matrix U to a second basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ by the formula

$$\mathbf{v}_j = \sum_i U_{ij} \mathbf{e}_i;$$

the matrix of action on ϕ on the new basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ will equal $U^{-1} F \phi(U)$, which is invertible if and only if F is.

The condition that $\Phi : \phi^* V \to V$ must be an isomorphism is the closest one can come to requiring the ϕ-action to be bijective without requiring that it be bijective on K itself. Here is when the latter happens.

Exercise 1.1.3. Prove that ϕ is bijective if and only if $\overline{\phi}$ is bijective.

Example 1.1.4. In the case where ϕ is the identity map, a ϕ-module is nothing more than a vector space equipped with an invertible linear transformation. If K were algebraically closed (and hence not discretely valued), we would get a decomposition into generalized eigenspaces. In the present case, one does at least get a direct sum decomposition of each

ϕ-module in which each summand splits over K^{alg} as a direct sum of generalized eigenspaces whose eigenvalues all have the same valuation. We will simulate this decomposition in the general case using the notion of a *pure* ϕ-module.

Example 1.1.5. There are also many interesting examples in arithmetic geometry of ϕ-modules for which ϕ is not the identity map. The usual source of these is the crystalline cohomology of schemes over a perfect field k of characteristic $p > 0$, or similar constructions such as the Dieudonné module of a p-divisible group. In these cases, K will be the fraction field of the ring $W(k)$ of Witt vectors of k; that is, $W(k)$ is the unique complete discrete valuation ring with maximal ideal (p) and residue field k. The endomorphism ϕ will be induced by the unique lift to $W(k)$ of the p-power Frobenius on k. (For instance, if $k = \mathbb{F}_p^{\mathrm{alg}}$, then K is the completion of the maximal unramified extension of $\mathbb{Q}_p$.)

Exercise 1.1.6. Write out the definitions of tensor products, symmetric powers, exterior powers, and duals in the category of ϕ-modules.

When ϕ is nontrivial, a ϕ-module does not have a well-defined determinant; however, this nonexistent determinant does at least have a well-defined valuation.

Definition 1.1.7. Let V be a ϕ-module over K of rank d. Let A be the matrix of action of ϕ on some basis of V, and define the *degree* of V as $\deg(V) = v(\det(A))$; it does not depend on the choice of the basis because for any $U \in \mathrm{GL}_d(K)$,

$$v(\det(U^{-1}A\phi(U))) = v(\det(U)^{-1}) + v(\det(A)) + v(\phi(\det(U))) = v(A).$$

Note that degree is additive: if $0 \to V_1 \to V \to V_2 \to 0$ is a short exact sequence of ϕ-modules over K, then $\deg(V) = \deg(V_1) + \deg(V_2)$. For V nonzero, define the *slope* of V to be the ratio $\mu(V) = \deg(V)/\operatorname{rank}(V)$.

Remark 1.1.8. The notion of slope is motivated by an analogy with the theory of vector bundles on an algebraic curve. Another analogous concept is the notion of *determinantal weight* used by Deligne in his second proof of the Weil conjectures [16].

Our best approximation to the notion of a ϕ-module having a single eigenvalue is the following.

Definition 1.1.9. Let V be a nonzero ϕ-module over K. We say V is *pure* if for some positive integer d and some basis $\mathbf{e}_1, \dots, \mathbf{e}_n$, the matrix of action A of ϕ^d on $\mathbf{e}_1, \dots, \mathbf{e}_n$ equals a scalar matrix times an element of $\mathrm{GL}_n(\mathfrak{o}_K)$; this forces $\mu(V) = \frac{1}{d}v(A)$. Another way to write this condition is $v(A) + v(A^{-1}) = 0$, where $v(A)$ denotes the minimum valuation of any entry of A. We say V is *étale* if it is pure of slope 0.

Example 1.1.10. Let π be any generator of $\mathfrak{m}_K$ (i.e., any uniformizer of K). For c, d integers with $d > 0$, define the ϕ-module $M_{\pi,c,d}$ of rank d to have its ϕ-action on a basis $\mathbf{e}_1, \dots, \mathbf{e}_d$ given by

$$\phi(\mathbf{e}_1) = \mathbf{e}_2, \quad \dots, \quad \phi(\mathbf{e}_{d-1}) = \mathbf{e}_d, \quad \phi(\mathbf{e}_d) = \pi^c \mathbf{e}_1.$$

Then $M_{\pi,c,d}$ is pure of slope $\frac{c}{d}v(\pi)$, because ϕ^d acts on $\mathbf{e}_1, \dots, \mathbf{e}_d$ via a diagonal matrix each of whose diagonal entries is the image of π^c under some power of ϕ. We usually only consider $M_{\pi,c,d}$ in case $\gcd(c,d) = 1$, in which case it will follow from Lemma 1.1.13 below that $M_{\pi,c,d}$ is irreducible.

Exercise 1.1.11. Suppose that V is pure, and that d is *any* integer for which $d\mu(V) = v(\lambda)$ for some $\lambda \in K$. Prove that there exists a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of V on which the matrix of action of ϕ^d equals λ times an element of $\mathrm{GL}_n(\mathfrak{o}_K)$. (The definition of purity only requires the *existence* of some such d.)

Exercise 1.1.12. Let $0 \to V_1 \to V \to V_2 \to 0$ be a short exact sequence of ϕ-modules, in which V_1, V_2 are pure *of the same slope*. Prove that V is also pure of that slope.

The following calculation shows that there are no maps between pure ϕ-modules of different slopes.

Lemma 1.1.13. *Let $\psi : V \to W$ be a nonzero homomorphism of pure ϕ-modules over K. Then $\mu(V) = \mu(W)$.*

Proof. Suppose first that $\dim(V) = 1$ and ψ is injective, so that we may identify V with a one-dimensional ϕ-stable subspace of W. Choose a positive integer d and a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of W on which ϕ^d acts via a matrix of the form λA with $\lambda \in K$ and $A \in \mathrm{GL}_n(\mathfrak{o}_K)$. Let $\mathbf{v}$ be a generator of V, and write $\mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$ and $\phi(\mathbf{v}) = b_1\mathbf{e}_1 + \cdots + b_n\mathbf{e}_n$. Then $a_1, \dots, a_n$ and $\lambda^{-1}b_1, \dots, \lambda^{-1}b_n$ generate the same $\mathfrak{o}_K$-submodule of K, since they can be written as $\mathfrak{o}_K$-linear combinations of each other using the entries of A and A^{-1}. On the other hand, since V is supposed to be ϕ-stable, we must have $\phi(\mathbf{v}) = \eta\mathbf{v}$ for some $\eta \in K$, which must then satisfy $\eta\lambda^{-1} \in \mathfrak{o}_K^\times$. Hence $\mu(V) = v(\eta) = v(\lambda)$ as claimed.

Suppose next that $\dim(V) = d > 0$ and ψ is injective. We may then replace V and W by $\wedge^d V$ and $\wedge^d W$ and apply the previous paragraph.

Suppose next that ψ is surjective. We may then replace V and W by $V^\vee$ and $W^\vee$ and apply the previous paragraph.

Suppose finally that ψ is arbitrary. We may then apply the preceding paragraphs to the maps $\ker(V) \to V$ and $V \to \mathrm{image}(V)$ to deduce the claim. □

Exercise 1.1.14. Let $0 \to V_1 \to V \to V_2 \to 0$ be a short exact sequence of ϕ-modules in which V_1, V_2 are pure and $\mu(V_1) > \mu(V_2)$. Prove that the sequence splits uniquely. (Hint: since the splitting is supposed to be unique, it is enough to find in the category of ϕ^d-modules for some d. A good warmup is to try the case where $\dim_K(V_1) = \dim_K(V_2) = 1$.)

The key structural result about ϕ-modules is the following.

Theorem 1.1.15. *Let V be an irreducible ϕ-module over K. Then V is pure.*

Proof. Let $K\{T\}$ be the *twisted polynomial ring* in K, in which the scalars commute with each other but satisfy the relation $Tr = \phi(r)T$ for $r \in K$. One may define Newton polygons for elements of $K\{T\}$, and prove a form of Hensel's lemma stating that any irreducible twisted polynomial has only one slope in its Newton polygon.

To apply this to the claim at hand, let $\mathbf{v} \in V$ be any nonzero element, and put $d = \dim_K V$. Then $\mathbf{v}, \phi(\mathbf{v}), \dots, \phi^{d-1}(\mathbf{v})$ span a nonzero ϕ-stable subspace of V, which must be all of V. Now consider the map $K\{T\} \to V$ sending $\sum_i a_i T^i$ to $\sum_i a_i \phi^i(\mathbf{v})$; its kernel is a left ideal of $K\{T\}$, which is principal by the Euclidean algorithm. We can thus write $V = K\{T\}/K\{T\}P$ for some twisted polynomial P, so that the ϕ-action is given by left multiplication by T.

If the polynomial P factored nontrivially as $P_1 P_2$, we would obtain a short exact sequence

$$0 \to P_1 K\{T\}/K\{T\}P \to K\{T\}/K\{T\}P \to K\{T\}/K\{T\}P_2 \to 0,$$

but such a sequence cannot exist because V is irreducible. Hence P is irreducible, and so has only one slope in its Newton polygon. From this, it is straightforward to check that V is pure. (See [29, §14.4] for a more detailed discussion.) □

This leads to the following description of an arbitrary ϕ-module.

Theorem 1.1.16. *Let V be a ϕ-module over K. There exists a unique filtration $0 = V_0 \subset \cdots \subset V_l = M$ by ϕ-submodules having the following properties.*

(i) For $i = 1, \dots, l$, the quotient V_i/V_{i-1} is pure.

(ii) We have $\mu(V_1/V_0) < \cdots < \mu(V_l/V_{l-1})$.

This filtration is called the slope filtration *of V. If $\overline{\phi}$ is bijective on k, then the slope filtration splits.*

Proof. We check only the existence, leaving the uniqueness as an exercise. The splitting in the case $\overline{\phi}$ is bijective on k will follow because ϕ is then also bijective (Exercise 1.1.3), so V may be viewed also in the category of ϕ^{-1}-modules. There, the slopes are all negated, so the steps of the slope filtration appear in the opposite order.

To check existence, suppose that we have any filtration which satisfies (i) but fails to satisfy (ii). There must then exist an index i for which $\mu(V_i/V_{i-1}) \geq \mu(V_{i+1}/V_i)$. If equality holds, then V_{i+1}/V_{i-1} is again pure by Exercise 1.1.12, so we may simply drop V_i from the filtration. If the inequality is strict, then the sequence

$$0 \to V_i/V_{i-1} \to V_{i+1}/V_{i-1} \to V_{i+1}/V_i \to 0$$

splits by Exercise 1.1.14. There then exists a ϕ-submodule of V_{i+1}/V_{i-1} which projects onto V_{i+1}/V_i, which we may write as V_i'/V_{i-1} for V_i' a ϕ-submodule of V. We may then replace V_i by V_i' in the filtration, in order to achieve $\mu(V_i'/V_{i-1}) < \mu(V_{i+1}/V_i')$.

Now start with any Jordan-Hölder filtration of V, i.e., any filtration whose successive quotients are irreducible. This satisfies (i) by Theorem 1.1.15. Repeat the argument in the previous paragraph (making arbitrary choices of i). We can only perform the first operation (omitting an index) as many times as the number of steps in the original filtration. Once we have exhausted the first operation, each instance of the second operation (changing the filtration at one position) decreases the number of pairs of indices $i < j$ for which $\mu(V_i/V_{i-1}) > \mu(V_j/V_{j-1})$. We must thus eventually run out of operations, at which point the filtration has the desired form. □

Exercise 1.1.17. Prove that the slope filtration of a ϕ-module is unique.

Exercise 1.1.18. Show that the slope filtration fails to split in the following example (in which $\overline{\phi}$ is not bijective): take K to be the completion of $\mathbb{Q}_p(t)$ for the Gauss norm (in which $v(t) = 0$), take ϕ to be the substitution $t \mapsto t^p$, and take V to be the ϕ-module on generators $\mathbf{e}_1, \mathbf{e}_2$ for which

$$\phi(\mathbf{e}_1) = \mathbf{e}_1, \qquad \phi(\mathbf{e}_2) = t + p\mathbf{e}_2.$$

For some k, one can classify ϕ-modules even further.

Definition 1.1.19. We say that k is *difference-closed* under $\overline{\phi}$ if every equation of the form $a_n\overline{\phi}^n(x) + \cdots + a_1\overline{\phi}(x) + a_0x = y$, with n a positive integer, $a_n \in k^\times$, and $a_{n-1}, \dots, a_0, y \in k$, has a nonzero solution $x \in k$. For instance, if k has characteristic $p > 0$ and $\overline{\phi}$ is a power of the absolute Frobenius, then k is difference-closed if and only if k is algebraically closed.

Theorem 1.1.20. *Suppose k is difference-closed under $\overline{\phi}$. Let π be any uniformizer of K. Let ϕ be any Frobenius lift on K. Then any ϕ-module over K is isomorphic to a direct sum in which each summand is an $M_{\pi,c,d}$ for some coprime integers c, d with $d > 0$.*

Proof. See [29, Theorem 14.6.3]. □

Remark 1.1.21. In the case where k is algebraically closed and $\overline{\phi}$ is the absolute Frobenius, Theorem 1.1.20 recovers the usual Dieudonné-Manin classification theorem.

Exercise 1.1.22. If k is difference-closed under $\overline{\phi}$, then Theorem 1.1.20 implies that every ϕ-module over K is completely reducible. Prove that this can fail if k is not difference-closed, even when $\overline{\phi}$ is bijective. (Hint: consider an extension of two étale ϕ-modules of rank 1 over $\mathbb{Z}_p$.)

Remark 1.1.23. When $\overline{\phi}$ is the absolute Frobenius (or a power thereof), there is a canonical way to extend K to a complete discretely valued field with algebraically closed residue field. The first step is to replace K by the completion K' of the direct limit

$$K \xrightarrow{\phi} K \xrightarrow{\phi} \cdots,$$

which has perfect residue field. Then K' can be written as a finite extension of $\operatorname{Frac} W(k^{\mathrm{perf}})$, and we may use Witt vector functoriality to form $K' \otimes_{W(k^{\mathrm{perf}})} W(k^{\mathrm{alg}})$.

For $\overline{\phi}$ general, it is possible to extend K and ϕ in order to force the residue field to become difference-closed, but not in a canonical way.

Exercise 1.1.24. Prove that there exists a complete field extension K' of K with the same value group, carrying an extension ϕ' of ϕ, such that the residue field of K' is perfect and difference-closed under the endomorphism induced by ϕ'. (Hint: use Zorn's lemma or an equivalent.)

Remark 1.1.25. One can generalize the argument in the proof of Theorem 1.1.15 to show that given any cyclic vector of V, i.e., any isomorphism $V \cong K\{T\}/K\{T\}P$ for a twisted polynomial P, the slopes of V are computed by the Newton polygon of P. In other words, if you have a basis of V on which ϕ acts via a *companion matrix*, then the slopes of V are computed by the characteristic polynomial of that matrix. However, it is not true that the slopes of V can be computed by taking the Newton polygon of the characteristic polynomial of the matrix of action on on an *arbitrary* basis; see [25, §1.3] for an example.

1.2 Slope filtrations over the Robba ring

In various applications (notably in p-adic Hodge theory, more on which later), one encounters ϕ-modules over somewhat more complicated rings than complete discretely valued fields. In the key case of the Robba ring, one can still construct slope filtrations.

Hypothesis 1.2.1. Throughout § 1.2, retain notation as in Hypothesis 1.1.1, but change the name of the endomorphism on K from ϕ to ϕ_K.

Definition 1.2.2. On $K[z, z^{-1}]$, extend the valuation v as the Gauss valuation:

$$v\left(\sum_i a_i z^i\right) = \min_i \{v(a_i)\};$$

For $r > 0$, also define the valuation w_r by the formula

$$w_r\left(\sum_i a_i z^i\right) = \inf_i \{v(a_i) + ir\};$$

for $\rho = e^{-r}$, we write $|\cdot|_\rho = \exp(-w_r(\cdot))$ for the corresponding Gauss norm.

Definition 1.2.3. For $r > 0$, let $\mathcal{R}_K^r$ be the Fréchet completion of $K[z, z^{-1}]$ for the valuations w_s for all $s \in (0, r]$. This gives the ring of rigid analytic functions on the annulus $0 < v(z) \leq r$; concretely, it may be identified with the set of formal sums $a = \sum_{i \in \mathbb{Z}} a_i z^i$ with $a_i \in K$ for which for each $s \in (0, r]$, $v(a_i) + is \to +\infty$ as $i \to \pm\infty$. Define the *Robba ring* $\mathcal{R}_K$ over K to be the union of the $\mathcal{R}_K^r$ over all $r > 0$; note that each element of $\mathcal{R}_K$ is a formal series which converges on some annulus of the form $0 < v(z) \leq *$, but there is no choice of a single annulus on which all of the elements of $\mathcal{R}_K$ converge.

There are various equivalent ways to formulate the growth conditions used to define the Robba ring.

Exercise 1.2.4. Prove that for any $r > 0$, a formal sum $a = \sum_i a_i z^i$ with coefficients in K belongs to $\mathcal{R}_K^r$ if and only if the following conditions both hold.

(i) We have $v(a_i) + ir \to +\infty$ as $i \to -\infty$.

(ii) For any $s > 0$, we have $v(a_i) + is \to +\infty$ as $i \to +\infty$.

Exercise 1.2.5. Prove that a formal sum $a = \sum_i a_i z^i$ with coefficients in K belongs to $\mathcal{R}_K$ if and only if

$$\liminf_{i \to -\infty} \frac{v(a_i)}{-i} > 0, \qquad \liminf_{i \to +\infty} \frac{v(a_i)}{i} \geq 0.$$

Remark 1.2.6. The ring $\mathcal{R}_K$ is inconvenient for certain purposes because it is not noetherian. However, by a result of Lazard [32], each of the $\mathcal{R}_K^r$ is a *Bézout domain*, i.e., an integral domain in which any finitely generated ideal is principal. It follows that $\mathcal{R}_K$ is also a Bézout domain. This is helpful because Bézout domains behave like principal ideal domains for many purposes; some of these are described in the following exercise.

Exercise 1.2.7. Let R be a Bézout domain.

(i) Prove that for any positive integer n, any $x_1, \dots, x_n \in R$ which generate the unit ideal appear as the first row of some invertible $n \times n$ matrix over R.

(ii) Prove that any finitely generated torsion-free module over R is free.

(iii) Let M be a finite free R-module. Prove that any saturated submodule of M is a direct summand of M. (A submodule N of M is *saturated* if M/N is torsion-free.)

Definition 1.2.8. Note that the theory of Newton polygons for (Laurent) polynomials over K extends to the Robba ring. An immediate consequence is that any unit of $\mathcal{R}_K$ belongs to the subring $\mathcal{R}_K^{\mathrm{bd}}$ consisting of series with bounded coefficients. Note that this subring is actually a discretely valued field under v which is not complete; however, see Exercise 1.2.9 below.

Note that v extends to a well-defined valuation on $\mathcal{R}_K^{\mathrm{bd}}$; we write $\mathcal{R}_K^{\mathrm{int}}$ to denote the valuation subring of $\mathcal{R}_K^{\mathrm{bd}}$, i.e., the series with coefficients in $\mathfrak{o}_K$. By contrast, a typical element of $\mathcal{R}_K$ with unbounded coefficients is

$$\log(1 + z) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} z^i,$$

and there is no well-defined p-adic valuation on such an element. One can define w_r for such an element for r sufficiently large (in this case any $r > 0$ will do), but these are not respected by the Frobenius lifts which we will consider shortly.

Exercise 1.2.9. Prove that the field $\mathcal{R}_K^{\mathrm{bd}}$, while not complete, is *henselian*. This implies for instance that it has a unique unramified extension for each finite separable extension of its residue field.

Definition 1.2.10. Let $q > 1$ be an integer. A *relative Frobenius lift* on $\mathcal{R}_K$ is an endomorphism ϕ of the form

$$\sum_i a_i z^i \mapsto \sum_i \phi_K(a_i)\phi(z)^i,$$

where ϕ_K is the isometric endomorphism of K fixed earlier, and $\phi(z)$ is an element of $\mathcal{R}_K$ such that $\phi(z) - z^q \in \mathfrak{m}_K \mathcal{R}_K^{\mathrm{int}}$. (If k is imperfect, one may prefer a slightly looser definition not requiring ϕ to carry K into itself, but we will not worry about this.)

In case k has characteristic $p > 0$ and q is a power of p, we may be interested in the special case where ϕ_K induces the q-power absolute Frobenius on k. In such a case, we call ϕ an *absolute Frobenius lift* on $\mathcal{R}_K$.

Remark 1.2.11. The definition of Frobenius lifts is the reason why we consider the Robba ring, rather than the ring of power series convergent on a *particular* annulus. Already in the case $\phi(z) = z^p$, applying ϕ does not preserve the inner radius of convergence of a series. One does however have some compatibility between ϕ and w_r, as in the following exercise.

Exercise 1.2.12. Let ϕ be any relative Frobenius lift on $\mathcal{R}_K$. Prove that if $r > 0$ satisfies $w_{r/q}(\phi(z)/z^q - 1) > 0$, then for all $x \in \mathcal{R}_K^r$,

$$w_{r/q}(\phi(x) - x^q) > w_r(x).$$

Consequently, for $r > 0$ sufficiently close to 0 (depending only on ϕ), for any $x \in \mathcal{R}_K^r$,

$$w_r(x) = w_{r/q}(\phi(x)).$$

(See [27, Lemma 2.3.3] and [28, Remark 1.2.5].)

Hypothesis 1.2.13. For the rest of § 1.2, fix a relative Frobenius lift ϕ on $\mathcal{R}_K$.

Definition 1.2.14. A *ϕ-module* over $\mathcal{R}_K^{\mathrm{bd}}$ or $\mathcal{R}_K$ is a finite free module M over that ring, equipped with an isomorphism $\Phi : \phi^* M \to M$. We again identify Φ with a semilinear ϕ-action that carries some (and hence any) basis of M to another basis.

Remark 1.2.15. The reader familiar with nonarchimedean analytic geometry may wonder whether we lose some generality in considering finite free modules over $\mathcal{R}_K$, rather than locally free coherent sheaves on an open annulus with outher radius 1. The answer is no: since K is discretely valued, any such sheaf is represented by a finite free module, i.e., it is generated by finitely many global sections. (In fact, this only requires K to be spherically complete. See for instance [26, Theorem 3.14].)

For the remainder of this lecture (with one exception; see Theorem 1.3.14), we will consider modules rather than sheaves. We will be forced to introduce the geometric viewpoint when we consider families, at which point we will develop it in more detail; see § 2.3.

With a bit of care, we can extend the notions of degree, slope, and purity to the setting of ϕ-modules over $\mathcal{R}_K^{\mathrm{bd}}$ or $\mathcal{R}_K$.

Definition 1.2.16. Let M be a nonzero ϕ-module over $\mathcal{R}_K^{\mathrm{bd}}$ or $\mathcal{R}_K$. The matrix of action A of ϕ on any basis of M has determinant which is a unit in $\mathcal{R}_K$. In particular, $\det(A) \in \mathcal{R}_K^{\mathrm{bd}}$ and $v(A)$ is well-defined. The same is true for the change-of-basis matrix between two bases of M, so we may again unambiguously define the *degree* and *slope* of M as $\deg(M) = v(\det(A))$ and $\mu(M) = \deg(M)/\operatorname{rank}(M)$.

We say M is *pure* if there exists a basis of M on which for some positive integer d, the matrix of action of ϕ^d equals a scalar in $\mathcal{R}_K^{\mathrm{bd}}$ times an invertible matrix over $\mathcal{R}_K^{\mathrm{int}}$; note that the scalar can always be taken within K. We again say M is *étale* if it is pure of slope 0.

The following key example from p-adic Hodge theory shows that Lemma 1.1.13 does not extend to this setting.

Example 1.2.17. Suppose $K = \mathbb{Q}_p$ and $\phi(z) = (z+1)^p - 1$, and put $t = \log(1+z)$. Then $\phi(t) = pt$, so $t\mathcal{R}_K$ is a ϕ-submodule (pure of slope 1) of the trivial ϕ-module $\mathcal{R}_K$ (pure of slope 0).

One does however have the following inequality, which resembles the situation of vector bundles on an algebraic curve (aside from a sign discrepancy).

Lemma 1.2.18. *Let $\psi : M \to N$ be a nonzero homomorphism of pure ϕ-modules over $\mathcal{R}_K$. Then $\mu(M) \geq \mu(N)$.*

Proof. See [29, Corollary 16.3.5]. □

Homomorphisms between pure modules of the same slope are particularly restricted.

Theorem 1.2.19. *Let M, N be pure ϕ-modules* of the same slope *over $\mathcal{R}_K^{\mathrm{bd}}$.*

(a) Any homomorphism from $M \otimes_{\mathcal{R}_K^{\mathrm{bd}}} \mathcal{R}_K$ to $N \otimes_{\mathcal{R}_K^{\mathrm{bd}}} \mathcal{R}_K$ is induced by a (unique) homomorphism from M to N. In particular, any pure ϕ-module over $\mathcal{R}_K$ descends uniquely to a pure ϕ-module over $\mathcal{R}_K^{\mathrm{bd}}$.

(b) Let $0 \to M \otimes_{\mathcal{R}_K^{\mathrm{bd}}} \mathcal{R}_K \to P \to N \otimes_{\mathcal{R}_K^{\mathrm{bd}}} \mathcal{R}_K \to 0$ be a short exact sequence of ϕ-modules over $\mathcal{R}_K$. Then P is also pure (of the same slope).

Proof. For (a), see [29, Corollary 16.3.4]. For (b), see [29, Proposition 16.3.9]. □

We have an analogue of the purity theorem for ϕ-modules over a field (Theorem 1.1.15), but its proof is significantly more complicated. We will return to it after we introduce the analogue of the Dieudonné-Manin decomposition over the Robba ring.

Theorem 1.2.20. *Any irreducible ϕ-module over $\mathcal{R}_K$ is pure.*

From this result, we derive consequences as in the case over a field.

Theorem 1.2.21. *Let M be a ϕ-module over $\mathcal{R}_K$. There exists a unique filtration $0 = M_0 \subset \cdots \subset M_l = M$ by saturated ϕ-submodules having the following properties.*

(i) *For $i = 1, \dots, l$, the quotient M_i/M_{i-1} is pure. (Note that M_i/M_{i-1} is free by Exercise 1.2.7, so it makes sense to view it as a ϕ-module.)*

(ii) *We have $\mu(M_1/M_0) < \cdots < \mu(M_l/M_{l-1})$.*

This filtration is called the slope filtration *of M.*

Proof. This follows from Theorem 1.2.20 as in the proof of Theorem 1.1.16. □

Corollary 1.2.22. *In Theorem 1.2.21, M is pure if and only if there does not exist a nonzero ϕ-submodule of M of slope less than $\mu(M)$.*

Remark 1.2.23. By analogy with the theory of vector bundles, one might characterize the latter condition in Corollary 1.2.22 by saying that M is *semistable*. However, this is not the same semistability occurring in the theory of p-adic Galois representations (though the two are distantly related).

Corollary 1.2.24. *In Theorem 1.2.21, the least slope achieved by a nonzero ϕ-submodule of M (not necessarily saturated) is $\mu(M_1)$. Moreover, any submodule achieving this slope is a saturated ϕ-submodule of M_1.*

Definition 1.2.25. We define the *slope multiset* of a ϕ-module M over $\mathcal{R}_K$ by constructing the slope filtration as in Theorem 1.2.21, then taking the multiset consisting of $\mu(M_i/M_{i-1})$ with multiplicity $\mathrm{rank}(M_i/M_{i-1})$ for $i = 1, \dots, l$. It is convenient to assemble these into an open convex polygon in the usual fashion: sort the slopes into increasing order, then use each slope in turn as the slope of a segment of the polygon whose horizontal projection has length equal to the multiplicity of that slope. The resulting polygon is sometimes called the *Newton polygon*, the *slope polygon*, or the *Harder-Narasimhan polygon* of M. (The last name is again motivated by the analogy with vector bundles.)

Exercise 1.2.26. Let M, N be ϕ-modules over $\mathcal{R}_K$ with slope multisets $s_1, \dots, s_m$ and $t_1, \dots, t_n$, respectively.

(i) Prove that the slope multiset of $M \oplus N$ consists of $s_1, \dots, s_m, t_1, \dots, t_n$. the slope multisets of M and N.

(ii) Prove that the slope multiset of $M \otimes N$ consists of $s_i + t_j$ for $i = 1, \dots, m, j = 1, \dots, n$.

(iii) Prove that for $i = 1, \dots, m$, the slope multiset of $\wedge^i M$ consists of $s_{j_1} + \cdots + s_{j_i}$ for all i-tuples $(j_1, \dots, j_i)$ of integers with $1 \leq j_1 < \cdots < j_i \leq m$.

(iv) Prove that the slope multiset of the dual $M^\vee$ of M consists of $-s_1, \dots, -s_m$.

Definition 1.2.27. Let M be a ϕ-module over $\mathcal{R}_K^{\mathrm{bd}}$. There are two natural ways to associate a slope multiset to M. One is by tensoring up to the completion of $\mathcal{R}_K^{\mathrm{bd}}$ under v, and using the slopes coming from Theorem 1.1.16; we call this the *generic slope multiset* of M. The other is by tensoring up to $\mathcal{R}_K$ and using the slopes coming from Theorem 1.2.21; we call this the *special slope multiset* of M.

The terminology comes from the case where M arises from a Dieudonné module (or F-crystal) over a local ring; in this case, the generic and special slopes are related to the base changes to the generic and special point. This assertion includes the following not entirely trivial fact: if M is obtained by base change from a finite free module over $\mathcal{R}_K^{\mathrm{bd}} \cap K[[t]]$, and M_0 is the reduction of that module modulo t, viewed as a ϕ-module over K, then the slope multiset of M_0 coincides with the special slope multiset of M. See [29, Definition 16.4.5].

As in the case of Dieudonné modules, one has the following semicontinuity property of generic and special slope filtrations.

Theorem 1.2.28. *Let M be a ϕ-module over $\mathcal{R}_K^{\mathrm{bd}}$. Form generic and special slope polygons associated to M with the same left endpoint. Then the special polygon lies on or above the generic polygon, and the right endpoints also coincide.*

Proof. See [29, Theorem 16.4.6]. □

Remark 1.2.29. It is a serious deficit in the theory of slope filtrations that there is currently no analogue of Theorem 1.2.21 available when the base field K is not discretely valued. The dependence on this hypothesis arises from the corresponding dependence in the Dieudonné-Manin theory over extended Robba rings; see § 1.4. We will run squarely into this difficulty in the second lecture.

1.3 Slope filtrations and p-adic Hodge theory

We now describe the link between ϕ-modules over the Robba ring and p-adic Galois representations.

Hypothesis 1.3.1. Throughout § 1.3, let K_0 be a finite unramified extension of $\mathbb{Q}_p$, equipped with the p-adic valuation v_p normalized so that $v_p(p) = 1$.

Definition 1.3.2. For $r > 0$, let $\mathbf{B}^{\dagger,r}_{K_0,\mathrm{rig}}$ be a copy of $\mathcal{R}^r_{K_0}$ with the series variable labeled by π, and put $\mathbf{B}^{\dagger}_{K_0,\mathrm{rig}} = \cup_{r>0} \mathbf{B}^{\dagger,r}_{K_0,\mathrm{rig}} \cong \mathcal{R}_{K_0}$. Let $\mathbf{B}^{\dagger}_{K_0}$ be the subring of $\mathbf{B}^{\dagger}_{K_0,\mathrm{rig}}$ corresponding to $\mathcal{R}^{\mathrm{bd}}_{K_0}$.

We equip $\mathbf{B}^{\dagger}_{K_0,\mathrm{rig}}$ with the absolute Frobenius lift ϕ (extending the unique absolute Frobenius lift on K_0) for which $\phi(\pi) = (\pi+1)^p - 1$. We also equip $\mathbf{B}^{\dagger,r}_{K_0,\mathrm{rig}}$ and $\mathbf{B}^{\dagger}_{K_0,\mathrm{rig}}$ with an action of the group $\Gamma = \mathbb{Z}_p^\times$, for which K_0 carries the action of Γ via the cyclotomic character, and $\gamma(\pi) = (\pi+1)^\gamma - 1$; this action is continuous for the Fréchet topology on $\mathbf{B}^{\dagger,r}_{K_0,\mathrm{rig}}$. Put $t = \log(1+\pi)$, so that $\phi(t) = pt$ and $\gamma(t) = \gamma t$.

It turns out that the finite unramified extensions of $\mathbf{B}^{\dagger}_{K_0}$ are closely related to finite (possibly ramified) extensions of K_0 itself. This construction is a special case of the Fontaine-Wintenberger theory of fields of norms [18, 43].

Definition 1.3.3. For $r > 0$ and n a positive integer such that $r > 1/(p^{n-1}(p-1))$, the elements of $\mathbf{B}^{\dagger,r}_{K_0,\mathrm{rig}}$ correspond to series in π which converge at $\zeta - 1$ whenever ζ is a primitive p^n-th root of unity. Moreover, t vanishes to order 1 at $\zeta - 1$. We thus have a well-defined homomorphism $\theta_n : \mathbf{B}^{\dagger,r}_{K_0,\mathrm{rig}} \to K_0(\mu_{p^n})[\![t]\!]$ with dense image. Note that we have a commutative diagram

$$\begin{array}{ccc} \mathbf{B}^{\dagger,r}_{K_0,\mathrm{rig}} & \xrightarrow{\phi} & \mathbf{B}^{\dagger,r/p}_{K_0,\mathrm{rig}} \\ \downarrow{\scriptstyle\theta_n} & & \downarrow{\scriptstyle\theta_{n+1}} \\ K_0(\mu_{p^n})[\![t]\!] & \longrightarrow & K_0(\mu_{p^{n+1}})[\![t]\!] \end{array} \tag{1.3.3.1}$$

in which the bottom arrow acts as the absolute Frobenius lift on K_0, fixes μ_{p^n}, and carries t to pt.

Let S be a finite unramified extension of $\mathbf{B}^{\dagger}_{K_0}$; note that the actions of ϕ and Γ extend uniquely to S. For some $r > 0$, we can write $S = S_r \otimes_{\mathbf{B}^{\dagger,r}_{K_0}} \mathbf{B}^{\dagger}_{K_0}$ for some finite $\mathbf{B}^{\dagger,r}_{K}$-algebra S_r which is unramified with respect to the w_r for all $r \in (0, r]$. For n a positive integer with $1/(p^{n-1}(p-1)) < r$, put $T_n = S_r \otimes_{\mathbf{B}^{\dagger,r}_{K_0},\theta_n} K_0(\mu_{p^n})$. Using the commutative diagram (1.3.3.1), we obtain homomorphisms $T_n \to T_{n+1}$ which are ϕ-equivariant for the absolute Frobenius lift on K_0 (fixing μ_{p^∞}). By Lemma 1.3.4 below, T_n is a field for n large, so the direct limit T of the T_n is a finite field extension of $K_0(\mu_{p^\infty})$. Moreover, it is canonically independent of the choices of r and S_r.

We now convert the correspondence $S \rightsquigarrow T$ into a map of Galois groups. We first identify the absolute Galois group $G_{k_0((\overline{\pi}))}$, for k_0 the residue field of K_0, with $\mathrm{Gal}((\mathbf{B}^{\dagger}_{K_0})^{\mathrm{unr}}/\mathbf{B}^{\dagger}_{K_0})$. Since the correspondence $S \rightsquigarrow T$ is sufficiently functorial to commute with automorphisms of S, we get an action of $\mathrm{Gal}((\mathbf{B}^{\dagger}_{K_0})^{\mathrm{unr}}/\mathbf{B}^{\dagger}_{K_0})$ on T whenever S is Galois over $\mathbf{B}^{\dagger}_{K_0}$. In particular, we also get an action on the compositum $\tilde{T}$ of the extensions of $K_0(\mu_{p^\infty})$ induced by all finite unramified Galois extensions of $\mathbf{B}^{\dagger}_{K_0}$. This allows us to identify $G_{k_0((\overline{\pi}))}$ with the quotient of $G_{K_0(\mu_{p^\infty})}$ by a closed subgroup, or in other words, to write down a surjective continuous homomorphism $G_{K_0(\mu_{p^\infty})} \to G_{k_0((\overline{\pi}))}$.

Lemma 1.3.4. *With notation as in Definition 1.3.3, the ring T_n is a field for n large.*

Proof. By enlarging K_0, we may reduce to the case where the residue field ℓ of S is totally ramified over $k_0((\overline{\pi}))$. Choose a uniformizer $\overline{\alpha}$ of ℓ and let $\overline{P}(T)$ be its minimal polynomial over $k_0((\overline{\pi}))$. This polynomial is Eisenstein, i.e., it is monic with all nonleading coefficients in $\overline{\pi}k_0[\![\overline{\pi}]\!]$ and constant term not divisible by $\overline{\pi}^2$. Let $P[T]$ be any monic lift of $\overline{P}(T)$; then $S \cong \mathbf{B}^{\dagger}_{K_0}[T]/(P(T))$. It thus suffices to note that for n large, the image of $P(T)$ under θ_n is an Eisenstein polynomial over $K_0(\mu_{p^n})$, and is hence irreducible. □

Theorem 1.3.5. *The homomorphism $G_{K_0(\mu_{p^\infty})} \to G_{k_0((\overline{\pi}))}$ is a homeomorphism of profinite groups.*

Proof. It suffices to check that the correspondence $S \mapsto T$ of Definition 1.3.3 is injective; this amounts to checking that every finite extension of $K_0(\mu_{p^\infty})$ is contained in some T.

It suffices to consider extensions of the form $K(\mu_{p^\infty})$ for some finite extension K of K_0, since this includes all finite extensions of $K_0(\mu_{p^\infty})$. There is no harm in replacing K_0 by an unramified subextension of $K(\mu_{p^\infty})$, so we may also assume that $K(\mu_{p^\infty})$ is totally ramified over K_0.

Put $K_{0,n} = K_0(\mu_{p^n})$ and $K_n = K(\mu_{p^n})$. Note that under the map $\theta_n : \mathbf{B}^{\dagger,r}_{K_0,\mathrm{rig}} \to K_{0,n}$ (for $r > 1/(p^{n-1}(p-1))$), π maps to a uniformizer of $K_{0,n}$. It follows that any element of $\mathfrak{o}_{K_{0,n}}$ can be lifted to some polynomial in π with coefficients in $\mathfrak{o}_{K_0}$. Moreover, any element in $\mathfrak{m}_{K_{0,n}}$ can be lifted to a polynomial in π with all coefficients in $\mathfrak{m}_{K_0}$, whereas any nonzero element of $\mathfrak{o}_{K_{0,n}}$ can be lifted by a polynomial with *not* all coefficients in $\mathfrak{m}_{K_0}$.

Choose a positive integer n such that $[K_n : K_{0,n}] = [K_\infty : K_{0,\infty}]$. Note that K_n is totally ramified over $K_{0,n}$. If K_n is tamely ramified over $K_{0,n}$, we can write $K_n = K_{0,n}[T]/(P(T))$ with $P(T) = T^m - a$ for some positive integer m not divisible by p and some $a \in \mathfrak{o}_{K_{0,n}}$. In this case, lift a to $\tilde{a} \in \mathfrak{o}_{K_0}[\pi]$ with not all coefficients in $\mathfrak{m}_{K_0}$; then we may take $\mathbf{B}^\dagger_K = \mathbf{B}^\dagger_{K_0}[T]/(\tilde{P}(T))$ for $\tilde{P}(T) = T^m - \tilde{a}$. This polynomial obviously has separable reduction modulo $\mathfrak{m}_{K_0}$, so has the desired effect.

If K_n is not totally ramified over $K_{0,n}$, then $m = [K_n : K_{0,n}]$ must be divisible by p. We can then write $K_n = K_{0,n}[T]/(P(T))$ with $P(T) = T^m + \sum_{i=0}^{m-1} a_i T^i$ for some positive integer m divisible by p and some $a_0, \dots, a_{m-1} \in \mathfrak{m}_K$. We can also ensure that there exists an index $j \in \{1, \dots, m-1\}$ not divisible by p such that $a_j \neq 0$. (If $P(T)$ does not have this property, then $P(T+p)$ does because its coefficient of T^{m-1} must be pm.) Choose lifts $\tilde{a}_0, \tilde{a}_j \in \mathfrak{o}_{K_0}[\pi]$ of a_0, a_j with not all coefficients in $\mathfrak{m}_{K_0}$, and lifts $\tilde{a}_i \in \mathfrak{o}_{K_0}[\pi]$ of a_i for $i \notin \{0, j\}$ with all coefficients in $\mathfrak{m}_{K_0}$, and then take $\mathbf{B}^\dagger_K = \mathbf{B}^\dagger_{K_0}[T]/(\tilde{P}(T))$ for $\tilde{P}(T) = T^m - \sum_{i=0}^{m-1} \tilde{a}_i T^i$. This polynomial has separable reduction modulo $\mathfrak{m}_{K_0}$ because the reduction has all roots nonzero while its derivative is equal to T^{j-1} times a nonzero scalar. Hence this construction has the desired effect. □

Definition 1.3.6. Let K be a finite totally ramified extension of K_0. Let K_0' be the maximal unramified subextension of $K(\mu_{p^\infty})$. By Theorem 1.3.5, $K(\mu_{p^\infty})$ corresponds to a finite unramified extension $\mathbf{B}^\dagger_K$ of $\mathbf{B}^\dagger_{K_0}$. Put $\mathbf{B}^\dagger_{K,\mathrm{rig}} = \mathbf{B}^\dagger_K \otimes_{\mathbf{B}^\dagger_{K_0}} \mathbf{B}^\dagger_{K_0,\mathrm{rig}}$. These rings admit unique extensions of the actions of ϕ and Γ. We may (noncanonically) identify the rings $\mathbf{B}^\dagger_K, \mathbf{B}^\dagger_{K,\mathrm{rig}}$ with $\mathcal{R}_{K_0'}, \mathcal{R}^{\mathrm{bd}}_{K_0'}$ by lifting a uniformizer of the residue field of $\mathbf{B}^\dagger_K$.

Let G_K be the absolute Galois group of K, let H_K be the absolute Galois group of $K(\mu_{p^\infty})$, and put $\Gamma_K = G_K/H_K = \mathrm{Gal}(K(\mu_{p^\infty})/K)$. The cyclotomic character χ gives an isomorphism of Γ_K with an open subgroup of $\mathbb{Z}_p^\times$; via χ, we get an action of Γ_K on $\mathbf{B}^\dagger_K$ and $\mathbf{B}^\dagger_{K,\mathrm{rig}}$. (Note that the rings depend only on $K(\mu_{p^\infty})$, whereas K itself is reflected by the choice of the subgroup Γ_K within $\mathbb{Z}_p^\times$.)

By a (ϕ, Γ_K)-*module* over $\mathbf{B}^\dagger_K$ or $\mathbf{B}^\dagger_{K,\mathrm{rig}}$, we will mean a ϕ-module equipped with a semilinear action of Γ_K which commutes with ϕ and is continuous for the Fréchet topology. We say a (ϕ, Γ_K)-module is *étale* if its underlying ϕ-module is étale.

Remark 1.3.7. For a (ϕ, Γ_K)-module over $\mathbf{B}^\dagger_K$, the continuity of the action of Γ_K is a somewhat delicate point. For the p-adic topology on $\mathbf{B}^\dagger_K$, the action of each $\gamma \in \Gamma_K$ is continuous. However, for the action to be continuous, the map $\Gamma_K \to \mathrm{Aut}(\mathbf{B}_K^\dagger)$ would have to be continuous, which it is not. If it were, then modulo any power of p the action of some open subgroup of Γ_K would be trivial; however, even modulo p the action of any nontrivial $\gamma \in \Gamma_K$ is nontrivial.

There are (at least) three natural topologies on $\mathbf{B}_K^\dagger$ for which the action of Γ_K is continuous. One is the Fréchet topology, i.e., the subspace topology from $\mathbf{B}_{K,\mathrm{rig}}^\dagger$. Another is the *weak topology*, in which a sequence converges if and only if it is p-adically bounded, and modulo any power of p converges in the π-adic topology; this is incomparable with the Fréchet topology. A third is the locally convex direct limit topology given by imposing the Fréchet topology on each $\mathbf{B}_{K,\mathrm{rig}}^{\dagger,r}$ defined by w_r and v_p; this is finer than both the weak topology and the Fréchet topology.

In general, it is unclear whether a Γ_K-action on a ϕ-module which is continuous for one of these topologies is also continuous for the others. The exception is when the ϕ-module is pure; see Theorem 1.3.8.

We now have the following description of p-adic Galois representations.

Theorem 1.3.8. *Let K be a finite totally ramified extension of K_0. The following categories are equivalent.*

(a) The category of continuous representations of G_K on finite-dimensional $\mathbb{Q}_p$-vector spaces.

(b) The category of étale (ϕ, Γ_K)-modules over the p-adic completion $\mathbf{B}_K$ of $\mathbf{B}_K^\dagger$.

(c) The category of étale (ϕ, Γ_K)-modules over $\mathbf{B}_K^\dagger$.

(d) The category of étale (ϕ, Γ_K)-modules over $\mathbf{B}_{K,\mathrm{rig}}^\dagger$.

Proof. The equivalence between (a) and (b) is due to Fontaine [17]; we describe the two functors here, and leave the verification that they are quasi-inverses as an exercise. Given a continuous action of G_K on a finite-dimensional $\mathbb{Q}_p$-vector space V, the associated étale (ϕ, Γ_K)-module is $D(V) = (V \otimes_{\mathbb{Q}_p} \widehat{\mathbf{B}_K^{\mathrm{unr}}})^{H_K}$; this has the expected dimension by Theorem 1.3.5 plus Hilbert-Noether Theorem 90. In the opposite direction, given an étale (ϕ, Γ_K)-module D over $\mathbf{B}_K$, it can be shown (as in the proof of Theorem 1.1.20) that $V(D) = (D \otimes_{\mathbf{B}_K} \widehat{\mathbf{B}_K^{\mathrm{unr}}})^{\phi=1}$ is a $\mathbb{Q}_p$-vector space of the expected dimension. This space evidently carries an action of H_K; it also carries an action of Γ_K. We claim that the group of automorphisms of $\widehat{\mathbf{B}_K^{\mathrm{unr}}}$ generated by H_K and Γ_K is homeomorphic to G_K, which will yield the G_K-action on $V(D)$.

To establish this claim, it is enough to check that for a finite Galois extension L of K, the group G of automorphisms of $\mathbf{B}_L$ generated by H_K and Γ_K is isomorphic to $\mathrm{Gal}(L(\mu_{p^\infty})/K)$ in a fashion compatible with further field extensions. In fact, it is equivalent to check the claim with $\mathbf{B}_L$ replaced by $\mathbf{B}_L^\dagger$; in this case, for n sufficiently large, G acts via θ_n on

$$\mathbf{B}_L^\dagger \otimes_{\mathbf{B}_{K_0}^\dagger} K_0(\mu_{p^\infty}) \cong L(\mu_{p^\infty}).$$

The image of G in $\mathrm{Gal}(L(\mu_{p^\infty})/K)$ contains $\mathrm{Gal}(L(\mu_{p^\infty})/K(\mu_{p^\infty}))$ (from the H_K-action) and surjects onto $\mathrm{Gal}(K(\mu_{p^\infty})/K)$ (from the Γ_K-action), and so is all of $\mathrm{Gal}(L(\mu_{p^\infty})/K)$. We thus have a surjection $G \to \mathrm{Gal}(L(\mu_{p^\infty})/K)$; this must be an isomorphism by Theorem 1.3.5.

The equivalence between (b) and (c) is the base extension from $\mathbf{B}_K^\dagger$ to $\mathbf{B}_K$. It is relatively easy to show that the restriction is fully faithful; essential surjectivity was established by Cherbonnier and Colmez [12]. One can extract a simpler proof from the work of Berger and Colmez [7].

The equivalence between (c) and (d), originally observed by Berger, is the base extension from $\mathbf{B}_K^\dagger$ to $\mathbf{B}_{K,\mathrm{rig}}^\dagger$. The fact that this is an equivalence of categories is an immediate consequence of Theorem 1.2.19. □

We can now describe some applications of slope filtrations to the construction of Galois representations. The desire to extend these applications to families of representations will motivate the discussion in the second and third lectures.

Remark 1.3.9. It is possible to give a complete classification of (ϕ, Γ_K)-modules over $\mathbf{B}_{K,\mathrm{rig}}^\dagger$ of rank 1, and of all possible extensions between two of these. For $K = \mathbb{Q}_p$, this was done by Colmez [13, §2]; the general case was carried out by Nakamura [37].

Using this classification, it is possible to show that for $0 \to M_1 \to M \to M_2 \to 0$ a short exact sequence of (ϕ, Γ_K)-modules over $\mathbf{B}_{K,\mathrm{rig}}^\dagger$, in which M_1 and M_2 are of rank 1, $\mu(M) = 0$, and $\mu(M_1) > \mu(M_2)$, the sequence splits if and only if M is not étale. To show this, one uses Theorem 1.2.21 to see that if M is not pure, it must have a unique saturated rank 1 ϕ-submodule N with $\mu(N) < \mu(M)$. Since N is unique, it is stable under Γ_K and hence a (ϕ, Γ_K)-submodule. The image P of N in M_2 cannot be zero, as otherwise N would be a submodule of M_1 in violation of Lemma 1.2.18. We must then have $P \cong N$, and the original short exact sequence must correspond to an element of the extension group $\mathrm{Ext}(M_2, M_1)$ whose image in $\mathrm{Ext}(P, M_1)$ is zero. However, the explicit calculaton of these extension groups shows that such an element cannot exist. See [13, Proposition 3.5] or [37, Theorem 3.4].

Remark 1.3.10. Calculations such as those in Remark 1.3.9 depend on the theory of Galois cohomology for (ϕ, Γ_K)-modules, as introduced by Herr and Liu. We will treat this topic in more detail in the second lecture.

Definition 1.3.11. Let K be a finite totally ramified extension of K_0. A *filtered ϕ-module* over K consists of a ϕ-module D over K_0 together with an exhaustive decreasing filtration $\mathrm{Fil}^\cdot D_K$ on $D_K = D \otimes_{K_0} K$ by K-subspaces. Note that we do not yet insist on any relationship between the ϕ-module structure and the filtration.

The multiset containing i with multiplicity $\dim_K(\mathrm{Fil}^i D_K / \mathrm{Fil}^{i+1} D_K)$ comprises the *Hodge-Tate weights* of the filtered ϕ-module. Define $t_N(D) = \deg(D)$. Define $t_H(D)$ to be the sum of the Hodge-Tate weights of D.

Definition 1.3.12. For D a filtered ϕ-module over K, a *filtered ϕ-submodule* of D is a ϕ-submodule D' of D equipped with an exhaustive decreasing filtration $\mathrm{Fil}^\cdot D'_K$ such that

$\mathrm{Fil}^i D'_K \subseteq \mathrm{Fil}^i D_K$ for all i. If this last inclusion is an equality, we say D' is a *saturated* filtered ϕ-submodule of D.

We say that D is *weakly admissible* if the following conditions hold.

(i) We have $t_N(D) = t_H(D)$.

(ii) For any filtered ϕ-submodule D' of D, we have $t_N(D') \geq t_H(D')$. (It suffices to check this for D' saturated.)

A theorem of Colmez-Fontaine puts filtered ϕ-modules in a natural correspondence with Galois representations. We make this correspondence explicit in terms of slope filtrations, following Berger [6].

Definition 1.3.13. Let M be a (ϕ, Γ)-module over $\mathbf{B}^\dagger_{K,\mathrm{rig}}$. A *modification* of M is a (ϕ, Γ)-module M' over $\mathbf{B}^\dagger_{K,\mathrm{rig}}$ equipped with an isomorphism $M[t^{-1}] \cong M'[t^{-1}]$.

For some r, we may realize M as the base extensions of a finite free module M_r over a finite extension S_r of $\mathbf{B}^{\dagger,r}_{K'_0,\mathrm{rig}}$ which is unramified with respect to w_s for all $s \in (0, r]$, such that Γ acts on M_r and ϕ carries M into $M_{r/p} = M_r \otimes_{\mathbf{B}^{\dagger,r}_{K'_0,\mathrm{rig}}} \mathbf{B}^{\dagger,r/p}_{K'_0,\mathrm{rig}}$. We may similarly realize M' as a finite free module M'_r over S_r. For n a sufficiently large positive integer, the isomorphism $M[t^{-1}] \cong M'[t^{-1}]$ induces an isomorphism

$$M_r \otimes_{S,\theta_n} K(\mu_{p^n})((t)) \cong M'_r \otimes_{S,\theta_n} K(\mu_{p^n})((t)). \tag{1.3.13.1}$$

Theorem 1.3.14. *Let K be a finite totally ramified extension of K_0. Let D be a filtered ϕ-module over K, and view $M(D) = D \otimes_{K_0} \mathbf{B}^\dagger_{K,\mathrm{rig}}$ as a (ϕ, Γ_K)-module over $\mathbf{B}^\dagger_{K,\mathrm{rig}}$.*

(a) There exists a unique modification $M'(D)$ of $M(D)$ such that for n a sufficiently large positive integer, the t-adic filtration on the right side of (1.3.13.1) *coincides with the filtration on the left side given by tensoring the t-adic filtration with the filtration provided by D.*

(b) The ϕ-module $M'(D)$ is étale if and only if D is weakly admissible.

Proof. It is apparent that $M'(D)$ exists and is unique in the category of coherent locally free sheaves on an open annulus with outer radius 1, equipped with actions of ϕ and Γ. However, by Remark 1.2.15, these are exactly the (ϕ, Γ)-modules over $\mathbf{B}^\dagger_{K,\mathrm{rig}}$. This yields (a). (For a more explicit approach, see [5, §3.1].)

For (b), one first calculates that the slope of $M(D)$ is $t_N(D) - t_H(D)$, so condition (i) of the definition of weak admissibility is equivalent to the condition $\mu(M(D)) = 0$. Moreover, if condition (ii) fails, we can produce a ϕ-submodule of $M(D)$ of negative slope, so $M(D)$ cannot be étale. Conversely, if $M(D)$ is not étale, then by Theorem 1.2.21, it contains a maximal ϕ-submodule of negative slope. Since this submodule is stable under the action of Γ_K, it corresponds to a filtered ϕ-submodule of D violating condition (ii). $\square$

Remark 1.3.15. The representations corresponding to filtered ϕ-modules are precisely those which are *crystalline*. We will not define this class of representations explicitly; it is in some sense the smallest reasonable class containing the p-adic étale cohomology of X_K for any smooth proper scheme X over $\mathfrak{o}_K$. For such a representation, the *crystalline comparison theorem* implies that the associated filtered ϕ-module is canonically isomorphic to the algebraic de Rham cohomology of X, equipped with the Hodge filtration and the action of Frobenius coming from the comparison between de Rham cohomology on the generic fibre and crystalline cohomology on the special fibre. There is an analogous setup for semistable schemes over $\mathfrak{o}_K$, in which the crystalline representations are replaced by the larger class of *semistable* representations, and the filtered ϕ-modules are replaced by filtered (ϕ, N)-modules (carrying the extra structure of a monodromy operator N which intertwines with ϕ in a suitable manner). For references, see for instance the introduction to [38].

Remark 1.3.16. A distinct but related construction of the Galois representations associated to filtered ϕ-modules has been given by Kisin [31], using a variant of (ϕ, Γ)-module theory in which the role of the cyclotomic extension $\mathbb{Q}_p(\mu_{p^\infty})$ is played by the (non-Galois) Kummer extension $\mathbb{Q}_p(p^{1/p^\infty})$. Kisin's construction has certain technical advantages in some situations, because it produces modules with a ϕ-action over a complete open unit disc, rather than a Robba ring (although the ϕ-action is not an isomorphism everywhere). This makes it particularly useful for studying integral structures on p-adic representations.

Remark 1.3.17. We would be remiss in neglecting to mention Berger's original motivation for working with (ϕ, Γ)-modules over the Robba ring: to prove Fontaine's conjecture that any de Rham representation is potentially semistable. Berger's proof reduces this problem to a theorem from p-adic differential equations, the p-adic local monodromy theorem of André, Mebkhout, and Kedlaya. The basic construction is to replace the action of the group Γ_K, viewed as a one-dimensional p-adic Lie group, by the action of its Lie algebra; we will use this construction later to study Galois cohomology (see Lemma 2.6.14). See [29, §20–21] for discussion of the p-adic local monodromy theorem, and [4] for the application to Galois representations.

What one sees from Berger's approach is that some invariants of a Galois representation of an analytic nature can be detected more easily on the side of (ϕ, Γ)-modules. For instance, if one starts with a p-adic étale cohomology group of a smooth proper scheme over $\mathfrak{o}_K$, Grothendieck's "mysterious functor" (constructed by Fontaine) converts this data into the p-adic (crystalline/rigid) cohomology of the same scheme. However, the latter can be read off immediately from the (ϕ, Γ)-module, as described in [4].

More recently, Colmez and others have discovered that even more subtle analytic invariants, such as those appearing in the study of p-adic L-functions, also can be constructed from (ϕ, Γ)-modules. Indeed, for $K = \mathbb{Q}_p$ one can construct a p-adic local Langlands correspondence for two-dimensional representations that interpolates the usual local Langlands correspondence; whether this can be done more generally is a rich topic of current research.

1.4 Dieudonné-Manin decompositions

It will be convenient to have, in addition to the theory of slope filtrations over the Robba ring, an analogue of the Dieudonné-Manin classification. We cannot hope to do this over the Robba ring itself, essentially because the residue field of $\mathcal{R}_K^{\mathrm{bd}}$ is not difference-closed. Instead, we must pass to a larger ring, whose construction can be carried out somewhat more generally. That extra level of generality will play a crucial role in the study of geometric families of (ϕ, Γ)-modules in the third lecture.

Hypothesis 1.4.1. Throughout § 1.4, continue to retain Hypothesis 1.1.1, again changing the name of the endomorphism from ϕ to ϕ_K. Unless otherwise specified, assume also that K is of characteristic 0, k is of characteristic $p > 0$, K is absolutely unramified (i.e., $\mathfrak{m}_K$ is the ideal generated by p), and k is perfect and difference-closed under $\overline{\phi}_K$. We will write v_p instead of v for the p-adic valuation on K, to help distinguish it from a second valuation v_ℓ which we will also consider.

We start with a construction parallel to the construction of the Robba ring.

Definition 1.4.2. Let ℓ be a perfect overfield of k which is complete for a real valuation v_ℓ trivial on k, and carries an extension $\overline{\phi}$ of $\overline{\phi}_K$. Let $W(\ell)$ be the ring of Witt vectors over ℓ; this ring admits a unique Frobenius lift ϕ compatible with both ϕ_K and $\overline{\phi}$, induced by functoriality of the Witt vector construction.

Each element of $W(\ell)$ has a unique representation as a convergent sum $\sum_{i=0}^{\infty} p^i [x_i]$, where $x_i \in \ell$ and $[x_i]$ is the *Teichmüller lift* of x_i, the unique lift of x_i admitting all p-power roots in $W(\ell)$. For any $r > 0$, let $\tilde{\mathcal{R}}_\ell^{\mathrm{int}}$ be the subset of $W(\ell)$ consisting of those sums $\sum_i p^i [x_i]$ for which $i + v_\ell(x_i) r \to +\infty$ as $i \to +\infty$. This turns out to be a subring of $W(\ell)$, and the function w_r on $\tilde{\mathcal{R}}^{\mathrm{int}}$ given by the formula

$$w_r \left(\sum_{i=0}^{\infty} p^i [x_i] \right) = \min_i \{ i + v_\ell(x_i) r \}$$

turns out to be a valuation. (We will give a detailed proof of this later; see Lemma 3.2.4.) For $\rho = e^{-r}$, we write $|\cdot|_\rho = \exp(-w_r(\cdot))$ for the corresponding "Gauss" norm. Note that the analogue of Exercise 1.2.12 is immediate: one has $\phi([\overline{x}]) = [\overline{\phi}(\overline{x})]$ for all $\overline{x} \in \ell$ (because $\phi([\overline{x}])$ admits all p-power roots of unity), so

$$|x|_\rho = |\phi(x)|_{\rho^{1/q}} \qquad (x \in W(\ell),\ \rho \in (0,1)).$$

For $r > 0$, define the ring $\tilde{\mathcal{R}}_\ell^r$ to be the Fréchet completion of $\tilde{\mathcal{R}}_\ell^{\mathrm{bd}} = \tilde{\mathcal{R}}_\ell^{\mathrm{int}}[\frac{1}{p}]$ for the valuations w_s for all $s \in (0, r]$. Define the *extended Robba ring* $\tilde{\mathcal{R}}_\ell$ to be the union of the $\tilde{\mathcal{R}}_\ell^r$. This turns out to be a Bézout domain, and the Frobenius ϕ on $W(\ell)$ (which carries $[x_i]$ to $[x_i]^p$) extends by continuity to $\tilde{\mathcal{R}}_\ell$.

We define ϕ-modules, degrees, slopes, and purity by analogy with $\mathcal{R}_K$.

Remark 1.4.3. Beware that an arbitrary element of $\tilde{\mathcal{R}}_\ell$ cannot in general be written as a doubly infinite sum $\sum_{i\in\mathbb{Z}} p^i[x_i]$. However, it is possible to present any element (non-canonically) in the form $\sum_{i\in\mathbb{Z}} p^i u_i$ in which each $u_i \in W(\ell)$ has the form $\sum_{j=0}^\infty p^i[u_{ij}]$ with $v_\ell(u_{i0}) \leq v_\ell(u_{ij})$ for all $j > 0$. (See for instance the errata to [27].)

Exercise 1.4.4. Prove the following analogue of the Hadamard three circles inequality. For $\rho, \sigma \in (0,1]$ and $t \in [0,1]$, put $\tau = \rho^t \sigma^{1-t}$. Then

$$|\cdot|_\tau \leq |\cdot|_\rho^c |f|_\sigma^{1-c}. \tag{1.4.4.1}$$

Example 1.4.5. Suppose that ℓ is the completed perfect closure of $k((z))$. We may then identify $\tilde{\mathcal{R}}_\ell$ with formal sums $\sum_{i\in\mathbb{Z}[1/p]} c_i z^i$ satisfying the same growth conditions as in the definition of the Robba ring, plus the following additional condition: for any $a, b \in \mathbb{R}$, the set of indices $i \leq a$ for which $v_p(c_i) \leq b$ has bounded denominators. If $\mathcal{R}_K$ carries a Frobenius lift ϕ for which $\phi(z) = z^p$, then $\mathcal{R}_K$ embeds into $\mathcal{R}_\ell$ by identifying the two elements both called z.

If the Frobenius lift on $\mathcal{R}_K$ carries another shape, then things are a bit less straightforward. Using ϕ, we construct the p-adic completion of the direct limit

$$\mathcal{R}_K^{\mathrm{int}} \xrightarrow{\phi} \mathcal{R}_K^{\mathrm{int}} \xrightarrow{\phi} \cdots .$$

This ring admits a map from $W(\ell)$ with $[\overline{x}]$ mapping to $\lim_{n\to\infty} \phi_S^{-n}(x^{q^n})$ for any $x \in \mathcal{R}_K^{\mathrm{int}}$. This map turns out to be an isomorphism; by mapping $\mathcal{R}_K^{\mathrm{int}}$ into the first term of the direct system, we form a map $\mathcal{R}_K^{\mathrm{int}} \to W(\ell)$. It further turns out that this map is an isometry with respect to the w_r for all r as in Exercise 1.2.12; see [27, Lemma 2.3.5] or [28, Proposition 2.2.6]. It thus extends by continuity to a map $\mathcal{R}_K^r \to \tilde{\mathcal{R}}_\ell^r$ for all r as in Exercise 1.2.12, and hence to a map $\mathcal{R}_K \to \tilde{\mathcal{R}}_\ell$.

Remark 1.4.6. One obtains a construction similar to Example 1.4.5 in case ℓ is a field of *Mal'cev-Neumann series*. That is, for Γ a totally ordered subgroup of $\mathbb{R}$, take the set of formal sums $\sum_{i\in\Gamma} c_i z^i$ whose support is well-ordered (contains no decreasing subsequence). This example figures prominently in [28].

The analogue of the Dieudonné-Manin classification in this context is the following result. (This implies a slope filtration theorem for ℓ arbitrary, which we leave to the reader to formulate.)

Theorem 1.4.7. *Suppose ℓ is difference-closed under $\overline{\phi}$. Then any ϕ-module over $\tilde{\mathcal{R}}_\ell$ is isomorphic to a direct sum in which each summand is an $M_{p,c,d}$ for some coprime integers c, d with $d > 0$.*

Proof. As in [28, Theorem 2.1.8]. (See also [29, Corollary 16.5.8].) □

We now relax Hypothesis 1.4.1 by dropping the condition that k be perfect and difference-closed. One could also make the following assertions without assuming K is of characteristic 0 and absolutely unramified, at the expense of having to complicate notation somewhat in the preceding statements.

Remark 1.4.8. It was noted earlier that Theorem 1.2.21 is proved via Theorem 1.4.7. Namely, given an embedding $\mathcal{R}_K \to \tilde{\mathcal{R}}_\ell$ with ℓ perfect and difference-closed under $\overline{\phi}$, any ϕ-module over $\mathcal{R}_K$ acquires a slope filtration over $\tilde{\mathcal{R}}_\ell$. One then uses faithful flat descent to push the slope filtration back down to $\mathcal{R}_K$; see [28, §3]. (In the case of an absolute Frobenius lift, one can use a Galois descent instead; see [27].)

It remains to construct an embedding $\mathcal{R}_K \to \tilde{\mathcal{R}}_\ell$. For this, apply Exercise 1.1.24 to replace K by a larger field K' carrying an extension of ϕ, whose residue field k' is perfect and difference-closed. Then apply Exercise 1.1.24 again to extend $k'((z))$ to a perfect difference-closed field ℓ. To embed $\mathcal{R}_K$ into $\tilde{\mathcal{R}}_\ell$, first argue as in Example 1.4.5 to perfect the residue field of $\mathcal{R}_K^{\mathrm{int}}$, then use Witt vector functoriality.

Remark 1.4.9. While there is little difficulty in extending Theorem 1.4.7 to cases where K is discretely valued but not absolutely unramified, it is not known how to extend to the case where K is an arbitrary complete discretely valued field.

The dependence on the discreteness hypothesis can be seen from the following brief summary of the proof of Theorem 1.4.7. One first constructs an auxiliary filtration of the given module and associates a slope polygon to it. One then makes a series of successive modifications to the filtration in order to raise the slope polygon. This process terminates after finitely many steps, at which point one can show that the filtration splits and gives the desired decomposition.

The difficulty in extending Theorem 1.4.7 to nondiscretely valued fields is a serious issue in the theory of arithmetic families of ϕ-modules. We will see this in the second lecture.

2 Arithmetic families of ϕ-modules

In this lecture, we consider families of ϕ-modules over a rigid analytic base space on which ϕ does not act. Our discussion is driven by potential applications in the study of families of p-adic Galois representations, such as those arising from p-adic modular forms.

2.1 Nonarchimedean analytic spaces

It will be convenient to use Berkovich's language of nonarchimedean analytic spaces, rather than Tate's language of rigid analytic spaces. We will avoid the full force of this theory, as developed in [8, 9], and instead concentrate on subspaces of an affinoid space. See [14] for an introduction.

Hypothesis 2.1.1. Throughout § 2.1, let K be any complete nonarchimedean field.

Definition 2.1.2. For $r_1, \dots, r_n > 0$, define the *generalized Tate algebra* of (poly)radius $(r_1, \dots, r_n)$ to be the ring

$$K\langle T_1/r_1, \dots, T_n/r_n\rangle = \left\{ \sum_{i_1,\dots,i_n=0}^{\infty} c_{i_1,\dots,i_n} T_1^{i_1} \cdots T_n^{i_n} \in K[\![T_1, \dots, T_n]\!] : |c_{i_1,\dots,i_n}| r_1^{i_1} \cdots r_n^{i_n} \to 0 \right\},$$

i.e., the completion of $K[T_1, \ldots, T_n]$ for the $(r_1, \ldots, r_n)$-Gauss norm. In Berkovich's theory, an *affinoid algebra* over K is a Banach algebra over K which can be written as a quotient of a generalized Tate algebra (equipped with the quotient norm).

This notion of an affinoid algebra is somewhat more expansive than the usual definition due to Tate, in which one only takes quotients of $K\langle T_1, \ldots, T_n\rangle$ (that is, one requires $r_1 = \cdots = r_n = 1$). This gives the same effect as restricting $r_1, \ldots, r_n$ to the divisible closure of the image of $K^\times$ under $|\cdot|$. Berkovich identifies these as *strictly affinoid* algebras, as will we.

Remark 2.1.3. A general affinoid algebra can be equipped with many different norms, coming from different presentations. These are all equivalent, in the sense that for any two such norms $|\cdot|_1, |\cdot|_2$, there exists $c > 0$ such that $|\cdot|_1 \leq c|\cdot|_2$. In general, there is no distinguished choice among these. However, on a reduced affinoid algebra, there is a unique minimal norm called the *spectral norm* $|\cdot|_{\mathrm{sp}}$. It can be computed from any other norm $|\cdot|$ by the formula $|x|_{\mathrm{sp}} = \lim_{s\to\infty} |x^s|^{1/s}$.

Definition 2.1.4. Let S be any commutative Banach algebra over K (e.g., an affinoid algebra). The *Gel'fand spectrum* (or *Berkovich spectrum*, or simply *spectrum*) of S, denoted $M(S)$, is the set of multiplicative seminorms on S which are bounded above by the specified norm on S. This set (as well as the topology on it; see Definition 2.1.7) depends only on the equivalence class of Banach norms on S, not on the specific norm chosen.

Remark 2.1.5. For S a strictly affinoid algebra, one has an analogue of the Nullstellensatz: for any maximal ideal $\mathfrak{m}$ of S, the residue field $S/\mathfrak{m}$ is finite over K [10, Corollary 6.1.2/3], and hence admits a unique extension of the norm on K. In this way, one obtains a natural map from the maximal spectrum $\mathrm{Spm}(S)$ to $M(S)$, but in general this map is far from surjective. For instance, if $S = K\langle T\rangle$, then $M(S)$ contains the ρ-Gauss norm for each $\rho \in (0, 1]$, none of which belongs to $\mathrm{Spm}(S)$. (The ρ-Gauss norm corresponds to a "generic point" of the closed disc of radius ρ centered at the origin in $\mathbb{A}^1_K$.)

Definition 2.1.6. For S an affinoid algebra over K and $x \in M(S)$, let $\mathcal{H}(x)$ denote the *residue field* of x. This field is constructed by forming the quotient of S by the kernel of x, inducing a true norm on this quotient, passing to the fraction field (which carries a unique extension of the norm), and then completing. This is a complete extension of K, but need not be finite over K unless $x \in \mathrm{Spm}(S)$.

For $f \in S$, we write $f(x)$ to mean the image of f in $\mathcal{H}(x)$. This allows us to write $x(f)$, the value at f of the seminorm defined by x, in the more suggestive form $|f(x)|$.

We now introduce a topology on $M(S)$.

Definition 2.1.7. For S a commutative Banach algebra over K, we topologize $M(S)$ with the coarsest topology under which for each $f \in S$, the evaluation map $M(S) \to [0, +\infty)$ taking α to $\alpha(f)$ is continuous. This coincides with the subspace topology if we embed $M(S)$ into $\prod_{f\in S}[0, |f|_S]$; in fact, the image is seen to be closed (i.e., the property of being a multiplicative seminorm is defined by closed conditions). Since the factors of the product are compact, $M(S)$ is compact by Tikhonov's theorem. Any bounded K-algebra homomorphism $S \to T$ of Banach algebras induces a map $M(T) \to M(S)$ by composition.

Lemma 2.1.8. *If $S \to T$ is an isometric homomorphism of commutative Banach algebras, then $M(T) \to M(S)$ is surjective.*

Proof. Using the isometric hypothesis, this reduces to the case where $S = K$ and T is nonzero, and the claim is that $M(T) \neq \emptyset$. For this, see [8, §1]. $\square$

Remark 2.1.9. A useful basis of open sets for the topology on $M(S)$ is given by the sets of the form

$$\{x \in M(S) : |f_1(x)| \in I_1, \dots, |f_m(x)| \in I_m\} \qquad (2.1.9.1)$$

for all positive integers m, all choices of $f_1, \dots, f_m \in S$, and all choices of open intervals $I_1, \dots, I_m$.

Definition 2.1.10. Let S be a (strictly) affinoid algebra over K. A *(strictly) affinoid subdomain* of $M(S)$ is a subset $U \subseteq M(S)$ for which there exists a bounded K-algebra homomorphism from S to another (strictly) affinoid algebra T over K, which is initial for the property that the image of $M(T)$ in $M(S)$ is contained in U. It then turns out that $M(T) = U$, and that U is closed in S. By an *affinoid neighborhood* of a point $x \in M(S)$, we will mean an affinoid subdomain of $M(S)$ which contains an open neighborhood of x.

For example, for any $f_1, \dots, f_m \in S$ and any closed intervals $I_1, \dots, I_m$ for which $I_j \cap [0, +\infty) \neq [0, 0]$ for any j, the set

$$\{x \in M(S) : |f_1(x)| \in I_1, \dots, |f_m(x)| \in I_m\}$$

is an affinoid subdomain of $M(S)$. An affinoid subdomain of this form is said to be *rational*; note that the rational subdomains are exactly the closures of the basic open sets, so every point of $M(S)$ has a neighborhood basis consisting of rational subdomains. A somewhat deeper fact (which we will not need) is the Gerritzen-Grauert theorem: every affinoid subdomain of $M(S)$ is a finite union of rational subdomains. (See [42] for a proof in the language and spirit of Berkovich's theory.)

For gluing constructions, one normally considers the G-topology defined by finite covers of affinoid spaces by affinoid subdomains. This is because of the following glueing property.

Theorem 2.1.11 (Kiehl)**.** *Let S be an affinoid algebra over X. Let $M(T_1), \dots, M(T_n)$ be a finite cover of $M(S)$ by affinoid subdomains. Suppose we are given the following data.*

(a) For $i = 1, \dots, n$, a finite T_i-module N_i.

(b) For $i, j = 1, \dots, n$, an isomorphism $\psi_{i,j} : N_i \otimes_{T_i} (T_i \otimes_S T_j) \cong N_j \otimes_{T_j} (T_i \otimes_S T_j)$, satisfying the cocycle condition.

Then there exists a unique (up to unique isomorphism) finite S-module N equipped with isomorphisms $N \otimes_S T_i \cong N_i$ inducing the $\psi_{i,j}$. Moreover, N maps bijectively to the subset of $\prod_i N \otimes_S T_i$ consisting of tuples (n_i) such that for any i, j, n_i and n_j have the same image in $N \otimes_S T_i \otimes_S T_j$.

Definition 2.1.12. Theorem 2.1.11 states on one hand that $M(T_i) \mapsto N \otimes_S T_i$ is a coherent sheaf for the G-topology, and on the other hand any coherent sheaf for the G-topology arises from a finite S-module. We will use this language in referring to finite S-modules as *coherent sheaves*. For $M(T)$ an affinoid subdomain of $M(S)$, we will also refer to the *restriction* of such a sheaf to $M(T)$, meaning the sheaf associated to $N \otimes_S T$.

The relationship between the G-topology and Berkovich's topology can be seen in the following lemma.

Lemma 2.1.13. *Let S be a (strictly) affinoid algebra over K. Then any open cover of $X = M(S)$ can be refined to a finite cover of X by (strictly) affinoid subdomains.*

Proof. Since X is compact, we may assume that the original cover consists of finitely many basic open subsets. Let U be an open set in the cover, of the form (2.1.9.1). For $\eta \in (0,1)$, define the open subset

$$U_\eta = \{x \in X : |f_i(x)| \in \eta I_i \cap \eta^{-1} I_i \quad (i = 1, \dots, m)\}.$$

Note that each I_i is the union of the $\eta I_i \cap \eta^{-1} I_i$ over all $\eta \in (0,1)$, so U is the union of the U_η over all $\eta \in (0,1)$. Again because X is compact, we can choose $\eta \in (0,1)$ so that the sets U_η, for U running over the cover, again cover X.

Let J_i be a closed interval with $\eta I_i \cap \eta^{-1} I_i \subseteq J_i \subseteq I_i$. In case X is strictly affinoid, we may force J_i to have endpoints which are norms of elements of K^{alg}. For U as above, the set

$$V_\eta = \{x \in X : |f_i(x)| \in J_i \quad (i = 1, \dots, m)\}$$

is a (strictly) affinoid subdomain of X, and the V_η form a finite cover of X. □

2.2 Relative annuli

We need an observation about relative annuli in nonarchimedean analytic geometry, made most naturally in Berkovich's language.

Hypothesis 2.2.1. Throughout § 2.2, let K be any complete nonarchimedean field.

Definition 2.2.2. For $I \subseteq [0, +\infty)$ an interval, let $A_K(I)$ be the annulus $|z| \in I$ within the affine z-line over K. When I is written out explicity, we usually drop the enclosing parentheses; for instance, if $I = [\alpha, \beta)$, we write $A_K[\alpha, \beta)$ instead of $A_K([\alpha, \beta))$.

Lemma 2.2.3. *Let X be an affinoid space over K. Let $I \subseteq [0, +\infty)$ be a closed interval. Let V be a vector bundle (coherent locally free sheaf) on $X \times_K A_K(I)$. Then there exists a finite open cover $U_1, \dots, U_n$ of X such that for each $i \in \{1, \dots, n\}$, the restriction of V to $U_i \times_K A_K(I)$ is freely generated by global sections.*

Proof. Note that for each Berkovich point $x \in X$ with residue field $\mathcal{H}(x)$, $x \times_K A_K(I) \cong A_{\mathcal{H}(x)}(I)$ is an affinoid space over $\mathcal{H}(x)$ whose coordinate ring is a principal ideal domain. Hence the restriction of V to $x \times_K A_K(I)$ is free; let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a basis.

Since $X \times_K A_K(I)$ is also an affinoid space, by Theorem 2.1.11, V is generated over $X \times_K A_K(I)$ by finitely many global sections $\mathbf{v}_1, \dots, \mathbf{v}_m$. We can thus choose $c_{ij} \in \mathcal{O}(x \times_K A_K(I))$ for which $\mathbf{e}_j = \sum_i c_{ij}\mathbf{v}_i$ as sections of V over $x \times_K A_K(I)$.

For any $\epsilon > 0$, we can choose $d_{ij} \in \mathcal{O}(X \times_K A_K(I))$ and $e_{ij} \in \mathcal{O}(X)$ so that $e_{ij}(x) \neq 0$ and $|c_{ij} - d_{ij}/e_{ij}|_x < \epsilon$. Given such a choice, put $e = \prod_{ij} e_{ij}$, and let $U \subseteq X$ be the complement of the zero locus of e. Put $\mathbf{e}'_j = \sum_i (d_{ij}/e_{ij})\mathbf{v}_i$ as a section of V over $U \times_K A_K(I)$, and define the $n \times n$ matrix A over $\mathcal{O}(x \times_K A_K(I))$ by the formula $\mathbf{e}'_j = \sum_i A_{ij}\mathbf{e}_i$. For ϵ sufficiently small, the supremum norm of $A - 1$ over $x \times_K A_K(I)$ is less than 1, so A is invertible; that is, $\mathbf{e}'_1, \dots, \mathbf{e}'_n$ form another basis of V over $x \times_K A_K(I)$. Fix a choice of such an ϵ hereafter.

Define the matrix B over $\mathcal{O}(X \times_K A_K(I))$ by $B_{ij} = (e/e_{ij})d_{ij}$. The common zero locus of the maximal minors of B is a closed analytic subspace of $X \times_K A_K(I)$ which by the previous paragraph does not meet $x \times_K A_K(I)$. It thus fails to meet $U' \times_K A_K(I)$ for some open neighborhood U' of x (since $x \times_K A_K(I)$ is compact). Over $U' \times_K A_K(I)$, $\mathbf{e}'_1, \dots, \mathbf{e}'_n$ generate V.

We conclude that for each $x \in X$, there exists an open neighborhood U of x in X such that V is freely generated by global sections over $U \times_K A_K(I)$. Since X is compact, this yields the claim. □

By Lemma 2.1.13, this yields the following corollary.

Corollary 2.2.4. *With notation as in Lemma 2.2.3, there exists a finite cover* $Y_1, \dots, Y_n$ *of* X *by affinoid subdomains, such that for each* $i \in \{1, \dots, n\}$*, the restriction of* V *to* $Y_i \times_K A_K(I)$ *is freely generated by global sections. Moreover, if* X *is strictly affinoid over* K*, each of the* Y_i *may be taken to be strictly affinoid over* K *as well.*

Remark 2.2.5. It would be interesting to extend Lemma 2.2.3 by replacing $A_K(I)$ by a product $A_K(I_1) \times_K \cdots \times_K A_K(I_n)$; this would be a nonarchimedean analytic version of the Quillen-Suslin theorem. In the case where X is strictly affinoid and each I_i equals either $[0,1]$ or $[1,1]$, the desired result has been established by Lütkebohmert [35]; although that proof predates the language of Berkovich spaces, it is similar in spirit to our proof of Lemma 2.2.3.

Remark 2.2.6. The question of finite generation of vector bundles becomes much more delicate when formulated over the product of an affinoid space with an *open* disc or annulus. We have discussed already the case where the base affinoid space is $M(K)$ itself (Remark 1.2.15). The general case is more complicated; however, some difficulties go away when we restrict to considering ϕ-modules, as we will see in the next part of this lecture.

2.3 ϕ-modules in families

We now introduce families of ϕ-modules over a base. We start out at a level of generality sufficient to cover both the arithmetic and geometric cases we will look at, then specialize to the arithmetic case to get some extra structural information.

Hypothesis 2.3.1. Throughout § 2.3, let K be a finite extension of $\mathbb{Q}_p$. Let S be an affinoid algebra over K, equipped with an endomorphism $\phi_S : S \to S$ which is isometric for the chosen norm $|\cdot|_S$ on S. Define $v_S(\cdot) = -\log|\cdot|_S$.

Definition 2.3.2. For $r > 0$, let $\mathcal{R}_S^r$ be the completed tensor product $\mathcal{R}_K^r \widehat{\otimes}_K S$ for the Fréchet topology on $\mathcal{R}_K^r$. In concrete terms, $\mathcal{R}_S^r$ consists of formal sums $\sum_{i\in\mathbb{Z}} c_i z^i$ for which for each $s \in (0, r]$, $v_S(c_i) + is \to +\infty$ as $i \to \pm\infty$. We may also view $\mathcal{R}_S^r$ as the ring of analytic functions on the product space $M(S) \times_K A_K(e^{-r}, 1)$. We call $\mathcal{R}_S = \cup_{r>0}\mathcal{R}_S^r$ the *Robba ring* over S. Let $\mathcal{R}_S^{\mathrm{bd}}$ be the subring of $\mathcal{R}_S$ consisting of series with bounded coefficients. (Note that these constructions do not depend on the choice of the norm $|\cdot|_S$.)

The ring $\mathcal{R}_S$ is even more badly behaved than $\mathcal{R}_K$, since we lose the Bézout property. As a result, we are forced to consider not just modules over $\mathcal{R}_S$, but sheaves over a corresponding geometric space, when talking about families of ϕ-modules. (This fulfills a promise made in Remark 1.2.15.)

Definition 2.3.3. Let q be a power of p. A *(q-power) Frobenius lift* on $\mathcal{R}_S$ is an endomorphism ϕ of the form $\sum_i c_i z^i \mapsto \sum_i \phi_S(c_i)\phi(z)^i$, where $\phi(z) \in \mathcal{R}_S$ is an element for which $\phi(z) - z^q$ has all coefficients of norm less than 1. If $\phi(z) \in \mathcal{R}_K$, we say ϕ is *split*.

As in Exercise 1.2.12, there exists some $\delta \in (0, 1)$ such that $|\phi(z)/z^q - 1|_{\delta^{1/q}} < 1$, and any $\rho \in [\delta, 1)$ then satisfies $|\cdot|_\rho = |\phi(\cdot)|_{\rho^{1/q}}$. In geometric terms, for any $\rho \in [\delta, 1)$, ϕ induces a finite (of degree q) étale surjective morphism from $A_K[\rho^{1/q}, 1)$ to $A_K[\rho, 1)$. We say that such a δ is *good* for ϕ.

Definition 2.3.4. Let ϕ be a Frobenius lift on $\mathcal{R}_S$. A *ϕ-module* over $\mathcal{R}_S^{\mathrm{bd}}$ or $\mathcal{R}_S$ is a finite locally free module over the appropriate ring, equipped with an isomorphism $\Phi : \phi^*M \to M$.

A *family of ϕ-modules* over $\mathcal{R}_S$ is a coherent locally free sheaf on $M(S) \times_K A_K[\delta, 1)$ for some $\delta \in (0, 1)$ which is good for ϕ, equipped with an isomorphism with its ϕ-pullback over some subspace of the form $M(S) \times_K A_K[\epsilon, 1)$ for some $\epsilon \in (0, 1)$. We consider these in the direct limit category as $\delta \to 1^-$; that is, morphisms between such objects can be defined over $M(S) \times_K A_K[\epsilon, 1)$ for any $\epsilon \in (0, 1)$. With this understanding, there is a base extension functor from ϕ-modules over $\mathcal{R}_S^{\mathrm{bd}}$ or $\mathcal{R}_S$ to families of ϕ-modules over $\mathcal{R}_S$.

Definition 2.3.5. Let $\mathcal{R}_S^{\mathrm{int}}$ be the subring of $\mathcal{R}_S$ consisting of series whose coefficients have norm at most 1. We say a ϕ-module over $\mathcal{R}_S^{\mathrm{bd}}$ or $\mathcal{R}_S$, or a family of ϕ-modules over $\mathcal{R}_S$, is *étale* if it is obtained by base extension from a finitely generated module M over $\mathcal{R}_S^{\mathrm{int}}$ equipped with an isomorphism $\phi^*M \to M$. In particular, an étale object must descend to an étale ϕ-module over $\mathcal{R}_S^{\mathrm{bd}}$, which we call an *étale model* of the original object.

Lemma 2.3.6. *The étale model of an étale ϕ-module over $\mathcal{R}_S^{\mathrm{bd}}$, or an étale family of ϕ-modules over $\mathcal{R}_S$, is unique if it exists.*

Proof. See [30, Proposition 6.5]. □

Corollary 2.3.7. *Let $\mathcal{E}$ be a family of ϕ-modules over $\mathcal{R}_S$. Let $M(S_1), \dots, M(S_n)$ be a finite covering of $M(S)$ by affinoid subdomains, on each of which ϕ acts. Then $\mathcal{E}$ is étale if and only if for $i = 1, \dots, n$, the restriction of $\mathcal{E}$ to a family of ϕ-modules over $\mathcal{R}_{S_i}$ is étale.*

Proof. This follows from Lemma 2.3.6 plus the fact that the algebras $\mathcal{R}_S^{\mathrm{bd}}$ obey an analogue of Kiehl's theorem, which permits the gluing of the étale models. For the latter, see [30, Proposition 3.10]. □

2.4 Arithmetic families

We now restrict to the case of interest in this lecture. Throughout § 2.4, continue to retain Hypothesis 2.3.1.

Definition 2.4.1. We say that a ϕ-module over $\mathcal{R}_S^{\mathrm{bd}}$ or $\mathcal{R}_S$, or a family of ϕ-modules over $\mathcal{R}_S$, is *arithmetic* if ϕ is split and acts as the identity on S. Note that this condition allows us to perform base changes along arbitrary maps from S to another affinoid algebra. By contrast, in cases where ϕ acts nontrivially on S, we can only perform base change along ϕ-equivariant maps.

Definition 2.4.2. Let $\mathcal{E}$ be an arithmetic family of ϕ-modules over $\mathcal{R}_S$. Since we may perform arbitrary base changes on arithmetic families, we would like to study the variation of the polygon of $\mathcal{E}$ as we specialize to various points $x \in M(S)$. Unfortunately, since Theorem 1.2.21 only applies to discretely valued fields, we cannot even define the slopes at an arbitrary point of $M(S)$, let alone say anything meaningful about them.

However, we can at least define what it means for $\mathcal{E}$ to be *étale* at a point $x \in M(S)$. To define this, choose $\delta \in (0,1)$ which is good for ϕ, such that $\mathcal{E}$ is defined over $M(S) \times_K A_K[\delta, 1)$, and the isomorphism $\phi^*\mathcal{E} \cong \mathcal{E}$ is defined over $M(S) \times_K A_K[\delta^{1/q}, 1)$. We then require the existence of an affinoid neighborhood U of x in $M(S)$ and a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of $\mathcal{E}$ over $U \times_K A_K[\delta, \delta^{1/q}]$ on which ϕ acts via a matrix F which is invertible over $\mathcal{R}^{\mathrm{int}}_{\mathcal{H}(x)}$.

We define the *étale locus* of $\mathcal{E}$ to be the set of $x \in M(S)$ at which $\mathcal{E}$ is étale.

Exercise 2.4.3. Check that the definition of étaleness does not depend on the choice of δ. Check also that in case $x \in \mathrm{Spm}(S)$, Definition 2.4.2 is consistent with the usual definition of étaleness for ϕ-modules over $\mathcal{R}_K$.

We conjecture the following about the structure of the étale locus.

Conjecture 2.4.4. *Let $\mathcal{E}$ be an arithmetic family of ϕ-modules over $\mathcal{R}_S$. Then the étale locus of $\mathcal{E}$ is locally closed, i.e., it is the intersection of an open set and a closed set in $M(S)$.*

Example 2.4.5. One cannot expect the étale locus to be either open or closed in general. On one hand, take $S = \mathbb{Q}_p\langle t \rangle$, and take $\mathcal{E}$ to be free on two generators $\mathbf{e}_1, \mathbf{e}_2$ satisfying

$$\phi(\mathbf{e}_1) = \mathbf{e}_2, \qquad \phi(\mathbf{e}_2) = \mathbf{e}_1 + p^{-1}t\mathbf{e}_2.$$

The étale locus in this case is the closed set $|t| \leq |p|$, which is not open.

On the other hand, one can form a universal family of extensions of two rank 1 (ϕ, Γ)-modules whose slopes are nonzero but sum to 0. By Remark 1.3.9, the fibre of this family over any rigid analytic point is étale if and only if the family is nonsplit. Hence one can find a sequence of points with étale fibres converging to the split extension, so the étale locus is not closed.

Conjecture 2.4.6. *Let $\mathcal{E}$ be a family of ϕ-modules over $\mathcal{R}_S$. Then $\mathcal{E}$ is étale if and only is $\mathcal{E}$ is étale at each point of $M(S)$.*

The only partial results to date on these conjectures are the following. This one is [34, Theorem 3.12]; it is stated in terms of ϕ-modules, but it can probably be extended to families of ϕ-modules without much extra work.

Theorem 2.4.7 (Liu)**.** *Let $\mathcal{E}$ be a ϕ-module over $\mathcal{R}_S$. Then for any $x \in \mathrm{Spm}(S)$, there is an affinoid neighborhood U of x in $M(S)$ such that for each $y \in U \cap \mathrm{Spm}(S)$, the slope polygon of $\mathcal{E}$ at y lies on or above the slope polygon of $\mathcal{E}$ at x (with the same endpoint).*

This one is [30, Theorem 7.4].

Theorem 2.4.8 (Kedlaya-Liu)**.** *Let $\mathcal{E}$ be a family of ϕ-modules over $\mathcal{R}_S$ which is étale at some $x \in \mathrm{Spm}(S)$. Then there exists an affinoid neighborhood of x in $M(S)$ over which $\mathcal{E}$ is étale.*

2.5 Families of (ϕ, Γ)-modules

The main reason to consider arithmetic ϕ-modules is that they receive an analogue of the (ϕ, Γ)-module functor from arithmetic families of Galois representations. (Although some of the results below assume a reduced affinoid base, we suspect that reducedness is not essential.)

Definition 2.5.1. Let K be a finite extension of $\mathbb{Q}_p$. Let K_0' be the maximal unramified subextension of $K(\mu_{p^\infty})$. Let S be an affinoid algebra over $\mathbb{Q}_p$. An *arithmetic (ϕ, Γ_K)-module* (resp. *arithmetic family of (ϕ, Γ_K)-modules*) over $\mathbf{B}^\dagger_{K,\mathrm{rig}} \widehat{\otimes}_{\mathbb{Q}_p} S$ is an arithmetic ϕ-module (resp. an arithmetic family of ϕ-modules) over $\mathbf{B}^\dagger_{K,\mathrm{rig}} \widehat{\otimes}_{\mathbb{Q}_p} S$ equipped with a continuous action of Γ_K commuting with ϕ. Such an object is *étale* if its underlying ϕ-module (resp. family of ϕ-modules) is étale.

Theorem 2.5.2 (Berger-Colmez)**.** *Let K be a finite extension of $\mathbb{Q}_p$. Let S be a reduced affinoid algebra over $\mathbb{Q}_p$. Then there exists a fully faithful functor from the category of finite locally free S-modules equipped with continuous actions of G_K, to the category of étale arithmetic (ϕ, Γ_K)-modules over $\mathbf{B}^\dagger_{K,\mathrm{rig}} \widehat{\otimes}_{\mathbb{Q}_p} S$. Moreover, the construction commutes with arbitrary base change on S.*

Proof. The hard part is to check the case where the original S-module admits a basis on which the Galois action is integral for the spectral norm on S. This is [7, Théorème 4.2.9]. The general case follows from this either by a gluing argument [30, Theorem 3.11] or by a short calculation involving Fitting ideals [11, Lemme 3.18]. □

When $S = \mathbb{Q}_p$, the functor in Theorem 2.5.2 reduces to the usual (ϕ, Γ)-module functor from Theorem 1.3.8, which is an equivalence of categories. However, even some very simple examples indicate that no such equivalence assertion can hold for more general base spaces. (This example is from [7, Remarque 4.2.10].)

Example 2.5.3 (Chenevier)**.** Take $K = \mathbb{Q}_p$ and $S = \mathbb{Q}_p\langle t, t^{-1}\rangle$. Form the étale arithmetic (ϕ, Γ_K)-module over $\mathbf{B}^{\dagger}_{K,\mathrm{rig}}\widehat{\otimes}_{\mathbb{Q}_p} S$ which is free on a single generator $\mathbf{v}$ and satisfies

$$\phi(\mathbf{v}) = t\mathbf{v}, \qquad \gamma(\mathbf{v}) = \mathbf{v} \quad (\gamma \in \Gamma_K).$$

This cannot belong to the essential image of the Berger-Colmez functor, otherwise it would admit a nonzero ϕ-invariant element over $\tilde{\mathbf{B}}^{\dagger}_{K,\mathrm{rig}}\widehat{\otimes}_{\mathbb{Q}_p} S$, which it does not. (The tensorand on the left is the extended Robba ring $\tilde{\mathcal{R}}_\ell$ for ℓ an algebraic closure of the residue field of $\mathbf{B}^{\dagger}_K$.) See also Remark 2.5.5.

On the other hand, one does have the following positive result.

Theorem 2.5.4 (Kedlaya-Liu)**.** *Let K be a finite extension of $\mathbb{Q}_p$. Let S, T be reduced affinoid algebras over $\mathbb{Q}_p$ equipped with a homomorphism $S \to T$ which is inner (i.e., $M(T)$ is an affinoid subdomain of $M(S)$ contained in the interior of $M(S)$). Let $\mathcal{E}$ be an étale arithmetic (ϕ, Γ_K)-module over $\mathbf{B}^{\dagger}_{K,\mathrm{rig}}\widehat{\otimes}_{\mathbb{Q}_p} S$. Then $\mathcal{E}\widehat{\otimes}_S T$ belongs to the essential image of the functor described in Theorem 2.5.2.*

Proof. See [30, Theorem 4.3]. □

Remark 2.5.5. One can describe an obstruction to inverting the functor of Theorem 2.5.2 in terms of residual representations. For instance, in Example 2.5.3, one recovers from $\mathcal{E}$ a family of representations over each residue disc. If $\mathcal{E}$ came from a global family of representations, these local families would have their mod p reductions all simultaneously trivialized upon replacing K by some finite extension. However, no such uniform choice is possible in this case. See [30] for further discussion.

2.6 Galois cohomology

As originally observed by Herr [23, 24], there is a close relationship between the Galois cohomology of a p-adic Galois representation and the structure of the associated (ϕ, Γ)-module. It is natural to attempt to extend this relationship to families, but in doing so one immediately encounters many open questions.

For technical reasons, we consider only (ϕ, Γ_K)-modules over a relative Robba ring rather than honest families. We leave the necessary modifications to handle families to the reader.

Hypothesis 2.6.1. Throughout § 2.6, let K be a finite extension of $\mathbb{Q}_p$. Let S be an affinoid algebra over $\mathbb{Q}_p$. Let D be an arithmetic (ϕ, Γ_K)-module over $\mathbf{B}^{\dagger}_{K,\mathrm{rig}}\widehat{\otimes}_{\mathbb{Q}_p} S$.

Definition 2.6.2. Suppose that Γ_K is procyclic; let γ be a topological generator. The *Herr complex* of D is the complex $C_{\phi,\gamma}(D)$ given by

$$0 \to D \to D \oplus D \to D \to 0$$

where the first D is placed in degree 0, the map $D \to D \oplus D$ is $x \mapsto ((\phi - 1)x, (\gamma - 1)x)$, and the map $D \oplus D \to D$ is $(x, y) \mapsto (\gamma - 1)x - (\phi - 1)y$. The cohomology of $C_{\phi,\gamma}(D)$

is canonically independent of the choice of γ; we simply denote it as $H^i(D)$ here. (This construction does not have a standard name; we call it the *Herr cohomology* of D, but it might also make sense to call it *Herr-Liu cohomology* or even *Galois cohomology*.)

Let E be another arithmetic (ϕ, Γ_K)-module over $\mathbf{B}^\dagger_{K,\mathrm{rig}} \widehat{\otimes}_{\mathbb{Q}_p} S$. One has cup product maps $\cup : H^i(D) \times H^j(E) \to H^{i+j}(D \otimes E)$; these are obvious to write down except for $i = j = 1$, in which case one puts

$$(x, y), (z, w) \mapsto y \otimes \gamma(z) - x \otimes \phi(w).$$

The only case where Γ_K is not procyclic is where $p = 2$ and $\Gamma_K \cong \mathbb{Z}_p^\times$. In this case, we take γ to be a generator of $\mathbb{Z}_p^\times/\{\pm 1\}$, and replace D with its invariants under the action of $-1 \in \mathbb{Z}_p^\times$ everywhere in the definition of the Herr complex.

Remark 2.6.3. The Herr complex may be viewed as computing the continuous group cohomology (or rather monoid cohomology) associated to the action of the monoid $\mathbb{Z}_{\geq 0} \times \Gamma_K$.

Theorem 2.6.4 (Herr, Liu)**.** *Let V be a finite-dimensional $\mathbb{Q}_p$-vector space equipped with a continuous action of G_K. Then the continuous Galois cohomology of V is functorially isomorphic to the cohomology of the Herr complex of the étale (ϕ, Γ_K)-module $D^\dagger_{\mathrm{rig}}(V)$ over $\mathbf{B}^\dagger_{K,\mathrm{rig}}$ associated to V. Moreover, this isomorphism is compatible with cup products.*

Proof. It was shown by Herr [23] that the Galois cohomology is computed by the cohomology of the Herr complex of the (ϕ, Γ_K)-module over $\mathbf{B}_K$ associated to V. The comparison of the latter with the cohomology of the Herr complex over $\mathbf{B}^\dagger_{K,\mathrm{rig}}$ was then made by Liu [33, Theorem 1.1]. □

If we consider not necessarily étale (ϕ, Γ_K)-modules, we still have a form of Tate local duality; this can be used for instance to study the Galois cohomology of trianguline representations.

Theorem 2.6.5 (Liu)**.** *The following hold for any D.*

(a) The Herr cohomology groups $H^i(D)$ are finite dimensional over K for $i = 0, 1, 2$.

(b) The Euler characteristic $\chi(D) = \sum_{i=0}^2 (-1)^i \dim_K H^i(D)$ equals $-[K : \mathbb{Q}_p] \operatorname{rank}(D)$.

(c) Let ω be the (ϕ, Γ_K)-module associated to the Galois representation $\mathbb{Q}_p(1)$ (i.e., the cyclotomic character). Then the cup product pairing

$$H^i(D) \times H^{2-i}(D^\vee \otimes \omega) \to H^2(D \otimes D^\vee \otimes \omega) \to H^2(\omega) \cong \mathbb{Q}_p$$

is perfect for $i = 0, 1, 2$.

Proof. See [33, Theorem 1.2]. □

Remark 2.6.6. Liu's proof does not include an independent proof of Tate local duality for Galois representations. However, Herr did give a proof of Tate duality using (ϕ, Γ_K)-modules; see [24].

The study of Herr cohomology in families is in its infancy, so many questions remain open. Here are a few of them, formulated as conjectures.

Conjecture 2.6.7. *The Herr cohomology groups $H^i(D)$ are finite S-modules.*

Conjecture 2.6.8. *The formation of the Herr cohomology groups $H^i(D)$ commutes with base change along a flat morphism $S \to T$ of affinoid algebras.*

Remark 2.6.9. Conjecture 2.6.8 is plausible because for any affinoid subdomain $M(T)$ of $M(S)$, the morphism $S \to T$ of affinoid algebras is flat. This does not imply the conjecture, however, because forming the base change of D involves a *completed* tensor product. However, Conjecture 2.6.8 follows from Conjecture 2.6.7; see [40, Proposition 3.7].

Remark 2.6.10. For the Galois cohomology of a family of p-adic representations, the analogue of Conjecture 2.6.7 has been proved by Bellaïche; see [3, §2.3]. The analogue of Conjecture 2.6.8 should follow by a similar argument.

These arguments do not suffice to imply the cases of Conjectures 2.6.7 and 2.6.8 on the side of Herr cohomology, because one still lacks an analogue of Theorem 2.6.4. In fact, it may be easier to first prove Conjecture 2.6.7 and then deduce a comparison theorem by invoking Theorem 2.6.4 pointwise.

Bellaïche has also established Conjecture 2.6.7 for a family of rank 1, not necessarily arising from a family of representations. Again, see [3].

Conjecture 2.6.11. *There exists a closed analytic subspace Z of $M(S)$ such that the formation of the Herr cohomology groups $H^i(D)$ commutes with any base change that factors through the inclusion of an affinoid subdomain of $M(S) \setminus Z$.*

Remark 2.6.12. Conjecture 2.6.11 (which we also expect to follow from Conjecture 2.6.7) should allow extension of Theorem 2.6.5 to families, by induction on the dimension of S. Compare [39, §6.1].

To conclude this lecture, we mention some techniques which may be helpful in establishing the above conjectures. One of these is an alternate description of Herr cohomology in terms of a one-sided inverse of ϕ.

Definition 2.6.13. Define the map $\psi : \mathbf{B}^{\dagger}_{K,\mathrm{rig}} \to \mathbf{B}^{\dagger}_{K,\mathrm{rig}}$ to be p^{-1} times the trace of ϕ. Extend by continuity to $\mathbf{B}^{\dagger}_{K,\mathrm{rig}} \widehat{\otimes}_{\mathbb{Q}_p} S$ using the identity map on S.

One similarly obtains a map $\psi : D \to D$ which is additive and satisfies $\psi(\phi(r)\mathbf{v}) = r\psi(\mathbf{v})$ for all $r \in \mathbf{B}^{\dagger}_{K,\mathrm{rig}}$ and all $\mathbf{v} \in D$. In particular, $\psi \circ \phi = \mathrm{id}_D$, so ψ is surjective and $D = \phi(D) \oplus \ker(\psi)$.

Lemma 2.6.14. *For any $\gamma \in \Gamma_K$ of infinite order, the kernel of $\gamma - 1$ on $\mathbf{B}^{\dagger}_{K,\mathrm{rig}} \widehat{\otimes}_{\mathbb{Q}_p} S$ equals $L \otimes_{\mathbb{Q}_p} S$, for L the subfield of K_0' fixed by γ.*

The proof uses Berger's differential operator, which also creates the link between p-adic Hodge theory and p-adic differential equations (see Remark 1.3.17).

Proof. By a calculation of Berger [4, Lemme 4.1], for γ sufficiently close to 1, the series

$$\frac{\log(\gamma)}{\log(\chi(\gamma))} = -\frac{1}{\log(\chi(\gamma))} \sum_{i=1}^{\infty} \frac{(1-\gamma)^i}{i}$$

converges to an operator d on D, which does not depend on the choice of γ. A further calculation [4, Lemme 4.2] shows that in fact

$$d = t(1+\pi)\frac{d}{d\pi}.$$

Let us specialize to the case $D = \mathbf{B}^{\dagger}_{K,\mathrm{rig}} \widehat{\otimes}_{\mathbb{Q}_p} S$. If x is a ring element for which $\gamma(x) = x$ for some γ of infinite order, then in fact $\gamma(x) = x$ for all γ in some open subgroup of Γ_K. Hence $d(x) = 0$ and so $\frac{dx}{d\pi} = 0$. If we write $\mathbf{B}^{\dagger}_{K,\mathrm{rig}} \widehat{\otimes}_{\mathbb{Q}_p} S$ as a Robba ring over $K_0' \otimes_{\mathbb{Q}_p} S$ in the series variable z, we must also have $\frac{dx}{dz} = 0$, but this forces $x \in K_0' \otimes_{\mathbb{Q}_p} S$. The desired result follows. □

We expect the following to hold as in [12, Lemma I.5.1], although with a bit more work required because the fixed subring of $\mathbf{B}^{\dagger}_{K,\mathrm{rig}} \widehat{\otimes}_{\mathbb{Q}_p} S$ under γ is no longer an integral domain, let alone a field.

Conjecture 2.6.15. *Suppose that γ is a topological generator of Γ_K. Then the map $\gamma - 1$ is injective on $D^{\psi=0}$.*

We expect the following to hold as in [12, Proposition II.6.1].

Conjecture 2.6.16. *Suppose that γ is a topological generator of Γ_K. Then the map $\gamma - 1$ is a surjection from $D^{\psi=0}$ to itself.*

Assuming these lemmas, one gets the following alternate description of Herr cohomology.

Corollary 2.6.17. *Suppose that γ is a topological generator of Γ_K. Let $C_{\psi,\gamma}(D)$ be the complex given by*

$$0 \to D \to D \oplus D \to D \to 0$$

where the first D is placed in degree 0, the map $D \to D \oplus D$ is $x \mapsto ((\psi-1)x, (\gamma-1)x)$, and the map $D \oplus D \to D$ is $(x,y) \mapsto (\gamma-1)x - (\psi-1)y$. Then the map from $C_{\phi,\gamma}(D) \to C_{\psi,\gamma}(D)$ given by the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D & \longrightarrow & D \oplus D & \longrightarrow & D & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle -\psi \oplus \mathrm{id}} & & \downarrow{\scriptstyle -\psi} & & \\
0 & \longrightarrow & D & \longrightarrow & D \oplus D & \longrightarrow & D & \longrightarrow & 0
\end{array}$$

is a quasi-isomorphism. (If Γ_K is not procyclic, one gets the same conclusion after modifying the construction as in Definition 2.6.2.)

Remark 2.6.18. Using ψ, it should be possible to follow Herr's arguments in [24] to establish Conjecture 2.6.7 in case D is étale. For $S = K$, Liu adds two ingredients to prove Theorem 2.6.5.

- Liu studies objects which resemble (ϕ, Γ_K)-modules except that rather than being locally free, they are killed by a power of t.
- Liu also makes a close analysis of objects of rank 1 (following Colmez).

Both of these steps should extend to families. What is missing is the connection between these special cases and the general results, using slopes. For instance, for $S = K$, to get finiteness of cohomology, one can start with an arbitrary module D, then replace it by a nonsplit extension by a rank 1 module (compare Remark 1.3.9), repeat until one gets a pure module of integer slope, then twist by a power of t to get an étale module. Since one has finiteness for the étale module, for its twist, and for the rank 1 modules, one gets finiteness for the original module.

Unfortunately, due to the difficulties associated with slope filtrations in families, we do not presently have the ability to make such arguments in families. For instance, given D, we do not have a method for forming an extension $0 \to D \to E \to F \to 0$ with F of rank 1, such that E is pure of some slope.

Remark 2.6.19. Herr cohomology in families is important for the construction and analysis of Selmer groups, particularly for nonordinary representations. See for example the work of Pottharst [39] and Bellaïche [3].

3 Geometric families of ϕ-modules

In this lecture, we consider families of ϕ-modules over a rigid analytic base space on which ϕ acts as a Frobenius lift. We will describe some results relevant to the study of Galois representations of étale fundamental groups, including an application to the study of Rapoport-Zink period domains.

Beware that results stated in this lecture should all be considered work in progress, since no reference is yet available. We plan to prepare a detailed manuscript later, in conjunction with Ruochuan Liu. It is our understanding that similar results (also covering the link to p-divisible groups, which we do not treat) have been obtained by Faltings, but as of this writing we do not have a reference for that approach either.

3.1 Geometric families of ϕ-modules

In order to talk about geometric families of ϕ-modules, we consider only a special class of base spaces. Throughout § 3.1, retain Hypothesis 2.3.1.

Definition 3.1.1. A reduced affinoid algebra S over K is said to have *good reduction* if it has the form $R \otimes_{\mathfrak{o}_K} K$ for Spf R a formal scheme which is smooth affine over $\mathfrak{o}_K$, equipped with the spectral norm. In this case, R may be characterized as the subring S° of S consisting of power-bounded elements, and $\mathfrak{m}_K R$ may be characterized as the ideal $S^{\circ\circ}$ of S° consisting of topologically nilpotent elements. Define the *reduction* of S as the quotient $\overline{S} = S^\circ/S^{\circ\circ}$.

Example 3.1.2. The simplest example of an affinoid algebra of good reduction is the Tate algebra $S = \mathbb{Q}_p\langle T_1, \dots, T_n\rangle$. In this case, the spectral norm coincides with the $(1, \dots, 1)$-Gauss norm, and S° is the completion of $\mathbb{Z}_p[T_1, \dots, T_n]$ for this norm (or equivalently, for the ideal (p)). Another useful example is $S = \mathbb{Q}_p\langle T_1, \dots, T_n, T_1^{-1}, \dots, T_n^{-1}\rangle$ again carrying the $(1, \dots, 1)$-Gauss norm, in which case S° is the completion of $\mathbb{Z}_p[T_1, \dots, T_n, T_1^{-1}, \dots, T_n^{-1}]$ for this norm.

Definition 3.1.3. Let S be an affinoid algebra over K of good reduction. We say that a ϕ-module over $\mathcal{R}_S^{\mathrm{bd}}$ or $\mathcal{R}_S$, or a family of ϕ-modules over $\mathcal{R}_S$, is *geometric* if the action of ϕ on S induces a power of the absolute Frobenius on $S^\circ/S^{\circ\circ}$. We do not require the action of ϕ on $\mathcal{R}_S$ to be split.

Remark 3.1.4. For geometric families, one can only perform base change along maps $S \to T$ in case T itself is equipped with a suitable Frobenius lift and the map is Frobenius-equivariant. For instance, one can only perform base change to a point if it is fixed by Frobenius, and such fixed points are indexed by the points of $\mathrm{Spm}(\overline{S})$ by Exercise 3.1.5 below. This suggests that we should be looking at the Berkovich spectrum not of S but of its reduction; this is the point of view we adopt in the next part of the lecture.

Exercise 3.1.5. Let S be an affinoid algebra over K of good reduction, and suppose ϕ_S induces a power of the absolute Frobenius on $\overline{S}$. Prove that every point of $\mathrm{Spm}(\overline{S})$ lifts uniquely to a point of $\mathrm{Spm}\, S$ fixed by ϕ_S. (One might call this lift the *Teichmüller lift* with respect to the map ϕ_S.)

Definition 3.1.6. For $\overline{e} \in \overline{S}$ and $e \in S^\circ$ a lift of $\overline{e}$, the completed localization $S\langle e^{-1}\rangle = S\langle t\rangle/(te - 1)$ carries an extension of ϕ, and is canonically independent of the choice of the lift e. We thus denote it by $S\langle\overline{e}\rangle^{-1}$.

3.2 A lifting construction

We would like to study the variation of slope filtrations in a geometric ϕ-module over $\mathcal{R}_S$. However, we have already seen that among the rigid analytic points of S, we can only perform base change to one lift of each closed point of $\overline{S}$ (the Teichmüller lift). In order to carry out more base changes, we need a richer version of the Teichmüller lift construction that applies to Berkovich analytic points also. We will later link these lifts to p-adic Hodge theory via a version of the field of norms construction.

Hypothesis 3.2.1. Throughout § 3.2, let R be an $\mathbb{F}_p$-algebra which is *perfect*, i.e., for which the p-power map is an isomorphism. Equip R with the *trivial norm*, for which $|r| = 0$ if $r = 0$ and $|r| = 1$ otherwise. We may then view R as a Banach algebra over the field $\mathbb{F}_p$ equipped with the trivial norm. Put $S = W(R)[\frac{1}{p}]$, where $W(R)$ is the p-typical Witt ring, viewed as a Banach algebra over $\mathbb{Q}_p$.

Remark 3.2.2. Note that $M(R)$ naturally contains not just $\mathrm{Spm}(R)$ but the whole prime spectrum $\mathrm{Spec}(R)$; namely, each prime ideal $\mathfrak{p}$ corresponds to the seminorm induced by the trivial norm on $\mathrm{Spec}(R)/\mathfrak{p}$.

Definition 3.2.3. Given $x \in M(R)$, define the function $\tilde{x} : S \to [0, +\infty)$ as follows. Given $f \in S$, since R is a perfect $\mathbb{F}_p$-algebra, we may write f as a sum $\sum_{i=m}^{\infty} [f_i] p^i$ for some $m \in \mathbb{Z}$ and some $f_i \in R$ (where $[f_i]$ indicates the Teichmüller lift). Moreover, this expression is unique up to adding or removing leading zeroes from the sequence $f_m, f_{m+1}, \dots$. Put

$$\tilde{x}(f) = \max_{i \geq m} \{p^{-i} x(f_i)\}.$$

The right side makes sense because $p^{-i} x(f_i)$ is bounded above by p^{-i} and so tends to 0 as $i \to +\infty$, forcing the maximum to exist.

Lemma 3.2.4. *For any $x \in M(R)$, the function $\tilde{x}$ is a multiplicative seminorm on S bounded above by the p-adic norm, so defines a point $\lambda(x) \in M(S)$.*

Proof. Since $x(r) \leq 1$ for all $r \in R$, $\tilde{x}$ is bounded above by the p-adic norm. To check that x is a seminorm, we first note that for $r_1, r_2 \in R$, we have $[r_1] + [r_2] = \sum_{i=0}^{\infty} [Q_i(r_1, r_2)^{1/p^i}] p^i$ for some homogeneous polynomials $Q_i(x, y) \in \mathbb{Z}[x, y]$ of degree p^i. It follows that $x(Q_i(r_1, r_2))^{1/p^i} \leq \max\{x(r_1), x(r_2)\}$, and so

$$\tilde{x}([r_1] + [r_2]) \leq \max_i \{x(r_1), x(r_2)\} = \max_i \{\tilde{x}([r_1]), \tilde{x}([r_2])\}. \tag{3.2.4.1}$$

Given two general elements $f, g \in S$, we can write $f = \sum_{i=m}^{\infty} [f_i] p^i$ and $g = \sum_{i=m}^{\infty} [g_i] p^i$ for some $m \in \mathbb{Z}$ and some $f_i, g_i \in R$. We then have

$$\begin{aligned} \tilde{x}(f + g) &\leq \max_i \{\tilde{x}([f_i] p^i + [g_i] p^i)\} \\ &\leq \max_i \{\max\{\tilde{x}([f_i] p^i), \tilde{x}([g_i] p^i)\}\} \\ &= \max_i \{\tilde{x}(f), \tilde{x}(g)\}. \end{aligned}$$

Hence $\tilde{x}$ is indeed a seminorm.

To check that $\tilde{x}$ is multiplicative, again take two general elements $f, g \in S$. Since $\tilde{x}$ is multiplicative on Teichmüller lifts, it is easy to check that $\tilde{x}(fg) \leq \tilde{x}(f)\tilde{x}(g)$; it remains to check for equality, which we need only do in case $\tilde{x}(f), \tilde{x}(g)$ are both positive. Choose the minimal indices j, k for which $[f_j] p^j$ and $[g_k] p^k$ attain their maximal values, define $f' = \sum_{i \geq j} [f_i] p^i$, $g' = \sum_{i \geq k} [g_i] p^i$, and observe that $\tilde{x}(f') < \tilde{x}(f)$, $\tilde{x}(g') < \tilde{x}(g)$. It follows that $\tilde{x}(f'g') = \tilde{x}(fg)$, so we may replace f, g by f', g' for the purposes of this calculation. That is, we may assume $f_i = 0$ for $i < j$ and $g_i = 0$ for $i < k$.

With this assumption, when we write $fg = \sum_i [h_i] p^i$, we have $h_i = 0$ for $i < j + k$, and $h_{j+k} = f_j g_k$. Hence $\tilde{x}(fg) \geq \tilde{x}(f)\tilde{x}(g)$, from which it follows that $\tilde{x}$ is multiplicative. $\square$

Definition 3.2.5. Given $y \in M(S)$, define the function $\overline{y} : R \to [0, +\infty)$ by the formula $\overline{y}(r) = y([r])$. Note that for $r \in R$ and $\tilde{r} \in W(R)$ lifting r, the sequence $\phi^{-n}(\tilde{r}^{p^n})$ converges to $[r]$ for the p-adic norm. It thus converges under y uniformly for all $y \in M(S)$; in particular, for n sufficiently large, we have $y(\phi^{-n}(\tilde{r}))^{p^n} = \overline{y}(r)$.

Lemma 3.2.6. *For $y \in M(S)$, the function $\overline{y}$ is a multiplicative seminorm on R bounded above by the trivial norm, so defines a point $\mu(y) \in M(S)$.*

Proof. Given $r, s \in R$, choose any $\tilde{r}, \tilde{s} \in W(R)$ lifting them. As in Definition 3.2.5, for n sufficiently large, we have $\overline{y}(t) = y(\phi^{-n}(\tilde{t}))$ for $(t, \tilde{t}) = (r, \tilde{r}), (s, \tilde{s}), (r + s, \tilde{r} + \tilde{s})$. We deduce that $\overline{y}$ is a seminorm from this observation plus the fact that y is a seminorm. Since $\overline{y}$ is evidently multiplicative, we have the desired result. □

Theorem 3.2.7. *The functions $\lambda : M(R) \to M(S)$ and $\mu : M(S) \to M(R)$ are continuous. Moreover, for any $x \in M(R), y \in M(S)$, we have $(\mu \circ \lambda)(x) = x$ and $(\lambda \circ \mu)(y) \geq y$. (The latter means that for any $f \in S$, $(\lambda \circ \mu)(y)(f) \geq y(f)$.)*

Proof. We first check that λ is continuous. It suffices to check that for any $f \in S$ and $\epsilon > 0$, the sets $\{x \in M(R) : \lambda(x)(f) > \epsilon\}$ and $\{x \in M(R) : \lambda(x)(f) < \epsilon\}$ are open in $M(R)$. Write $f = \sum_{i=m}^{\infty} [f_i] p^i$, and choose j for which $p^{-j} < \epsilon$; then $\lambda(x)([f_i]p^i) < \epsilon$ for all $x \in M(R)$ and all $i \geq j$. We thus have

$$\{x \in M(R) : \lambda(x)(f) > \epsilon\} = \bigcup_{i=m}^{j-1} \{x \in M(R) : x(f_i) > p^i \epsilon\}$$

$$\{x \in M(R) : \lambda(x)(f) < \epsilon\} = \bigcap_{i=m}^{j-1} \{x \in M(R) : x(f_i) < p^i \epsilon\},$$

so both sets are open.

We next check that μ is continuous. It suffices to check that for any $r \in R$ and $\epsilon > 0$, the sets $\{y \in M(S) : \mu(y)(r) > \epsilon\}$ and $\{y \in M(S) : \mu(y)(r) < \epsilon\}$ are open in $M(S)$. However, these sets can also be defined as $\{y \in M(S) : y([r]) > \epsilon\}$ and $\{y \in M(S) : y([r]) < \epsilon\}$, in which form they are manifestly open.

The equality $(\mu \circ \lambda)(x) = x$ is evident from the definitions. The inequality $(\lambda \circ \mu)(y) \geq y$ follows from the definition of λ and the observation that $(\lambda \circ \mu)(y)([r]) = y([r])$ for any $r \in R$. □

Remark 3.2.8. Note that we distinguish points $x, y \in M(R)$ even if they define equivalent seminorms, i.e., even if $x = y^c$ for some $c > 0$. This is important because equivalent seminorms do not have equivalent images under λ. This is one of several reasons why it is better to consider multiplicative seminorms than Krull valuations in this discussion.

Here is a simple example to illustrate that $\lambda \circ \mu$ need not be the identity map.

Example 3.2.9. Put $R = \cup_{n=1}^\infty \mathbb{F}_p[X^{p^{-n}}]$. Put $S_0 = \mathbb{Q}_p\langle T \rangle$; we may then identify S with the completion of the direct limit of the system

$$S_0 \xrightarrow{\phi} S_0 \xrightarrow{\phi} \cdots .$$

Under this identification, T may be identified with the Teichmüller lift $[X]$.

Let $y_0 \in M(S_0)$ be the seminorm on S_0 factoring through $S_0/(T-p)S_0$. This seminorm extends uniquely to $y \in M(S)$ because the polynomials $T^{p^n} - p$ are all irreducible over $\mathbb{Q}_p$.

We now compute $\mu(y)$. Note first that $\mu(y)(X) = y(T) = p^{-1}$ and that $\mu(y)(r) = 1$ for $r \in \mathbb{F}_p^\times$. These imply that $\mu(y)(r) \leq p^{-p^{-n}}$ whenever $r \in \mathbb{F}_p[X^{1/p^n}]$ is divisible by $X^{p^{-n}}$, so $\mu(y)(r) = 1$ whenever $r \in \mathbb{F}_p^\times + X^{p^{-n}}\mathbb{F}_p[X^{p^{-n}}]$. We conclude that for $r \in R$, $\mu(y)(r)$ equals the X-adic norm on R with the normalization $\mu(y)(X) = p^{-1}$. In particular, we have a strict inequality $(\lambda \circ \mu)(y) > y$.

3.3 Lifting for Robba rings

We now adapt the lifting construction from the previous section to the relative Robba rings over which we are defining geometric families of ϕ-modules.

Hypothesis 3.3.1. Throughout § 3.3, let K be a finite *unramified* extension of $\mathbb{Q}_p$ equipped with its unique p-power Frobenius lift, with residue field k. Let S be an reduced affinoid algebra over K of good reduction. Let $\mathcal{E}$ be a geometric family of ϕ-modules over $\mathcal{R}_S$. Let $\delta_0 \in (0,1)$ be any value which is good for ϕ, such that $\mathcal{E}$ can be defined as a coherent locally free sheaf over $M(S) \times_K A_K[\delta_0, 1)$, and the ϕ-action is an isomorphism over $M(S) \times_K A_K[\delta_0^{1/q}, 1)$; note that the same properties hold if δ_0 is replaced by any $\delta \in [\delta_0, 1)$.

Definition 3.3.2. Equip k and $\overline{S}$ with the trivial norm. Fix a choice of $\omega \in (0,1)$, and equip $k' = k((\overline{z}))$ and $\overline{S}' = \overline{S}((\overline{z})) = \overline{S}\widehat{\otimes}_k k'$ with the z-adic norm normalized by $|\overline{z}| = \omega$.

Let $\overline{S}^{\mathrm{perf}}$ and $\overline{S}^{\prime,\mathrm{perf}}$ be the completed perfect closures of $\overline{S}$ and $\overline{S}'$, respectively. As in Example 1.4.5, the Frobenius lift ϕ_S defines a map from S° into the Witt ring $W(\overline{S}^{\mathrm{perf}})$. Similarly, the Frobenius lift ϕ defines a map from $\mathcal{R}_S^{\mathrm{int}}$ into $W(\overline{S}^{\prime,\mathrm{perf}})$.

Definition 3.3.3. Given a point $x \in M(\overline{S}') = M(\overline{S}^{\prime,\mathrm{perf}})$ and a quantity $\rho \in (0,1)$, apply the map λ from Lemma 3.2.4 to $x^{\log_\omega \rho} \in M(\overline{S}')$ to produce a multiplicative seminorm on $W(\overline{S}')$. For $\rho = e^{-r}$, this seminorm extends by continuity to a multiplicative seminorm $\lambda(x,\rho)$ on $\mathcal{R}_S^r$. One obtains the same function by restricting the map $|\cdot|_\rho$ on $\tilde{\mathcal{R}}_{\kappa_x}^r$, for κ_x the completion of the perfect closure of $\mathcal{H}(x)$, along the map $\mathcal{R}_S^r \to \tilde{\mathcal{R}}_{\kappa_x}^r$ induced by ϕ as in Example 1.4.5.

Lemma 3.3.4. *Choose any $x \in M(\overline{S}')$.*

(a) For any $f \in S^\circ((z))$ with $x(\overline{f}) > 0$, there exists $\delta \in [\delta_0, 1)$ such that for $\rho \in [\delta, 1)$, $\lambda(x,\rho)(f) = \lambda(x,\rho)([\overline{f}])$.

(b) For all $\rho \in [\delta_0, 1)$, $\lambda(x,\rho)(z) = \lambda(x,\rho)([\overline{z}]) = \rho$.

Proof. For (a), note that as ρ tends to 1, $\lambda(x,\rho)([\overline{f}]) = x(f)^{\log_\omega \rho}$ tends to 1 while $\lambda(x,\rho)(f - [\overline{f}])$ is bounded above by p^{-1}. Hence there exists $\delta \in [\delta_0, 1)$ such that for $\rho \in [\delta, 1)$, $\lambda(x,\rho)(f - [\overline{f}]) < \lambda(x,\rho)([\overline{f}])$ and so $\lambda(x,\rho)(f) = \lambda(x,\rho)([\overline{f}])$.

For (b), note that for arbitrary $\rho \in [\delta_0, 1)$, we have by Exercise 1.2.12

$$\lambda(x,\rho)(z) = \lambda(x,\rho^{1/q})(\phi(z)) = \lambda(x,\rho^{1/q})(z^q)$$
$$\lambda(x,\rho)([\overline{z}]) = \rho = \lambda(x,\rho^{1/q})([\overline{z}])^q.$$

Hence if the desired result holds for $\rho \in [\delta, 1)$, it also holds for $\rho \in [\delta', 1)$ for $\delta' = \max\{\delta_0, \delta^q\}$. This yields the desired result. □

Definition 3.3.5. For $x \in M(\overline{S}')$ and $\rho \in [\delta_0, 1)$, by Lemma 3.3.4, we may identify $\lambda(x,\rho)$ with a point of $M(S) \times_K A_K[\rho,\rho]$. These points have the property that

$$\phi(\lambda(x,\rho)) = \lambda(x,\rho^q) \qquad (x \in M(\overline{S}'),\ \rho \in [\delta_0^{1/q}, 1)).$$

Remark 3.3.6. Beware that a point of $M(S) \times_K A_K[\delta_0, 1)$ is in general *not* determined by its projections onto the two factors. This is because the fibred product in the category of nonarchimedean analytic spaces is formed by taking completed tensor products at the level of rings; as in the case of schemes, this construction is not compatible with the fibred product in the category of sets. This is relevant for the following exercise.

Exercise 3.3.7. Suppose that $x \in \mathrm{Spm}(\overline{S})$ and $y \in M(\overline{S}')$ is the ω-Gauss seminorm with respect to x. Check that for $\rho \in [\delta_0, 1)$, $\lambda(y,\rho)$ is the ρ-Gauss seminorm with respect to the Teichmüller lift of x (see Exercise 3.1.5).

3.4 Variation of slope filtrations

We are now ready to consider the variation of slope filtrations in a geometric ϕ-module over $\mathcal{R}_S$, by adapting arguments introduced by Hartl in an analogous equal-characteristic situation [20]. Throughout § 3.4, retain Hypothesis 3.3.1.

Definition 3.4.1. For $x \in M(\overline{S}')$, we write $\mathcal{E}_x$ for the base extension of $\mathcal{E}$ to $\tilde{\mathcal{R}}_{\kappa_x}$, and define the *slope polygon* of $\mathcal{E}$ at x as the slope polygon of $\mathcal{E}_x$. In particular, we say $\mathcal{E}$ is *étale* at x if $\mathcal{E}_x$ is étale, and we refer to the set of $x \in M(\overline{S}')$ at which $\mathcal{E}$ is étale as the *étale locus* of $\mathcal{E}$.

In contrast to the arithmetic case, one has fairly good control over the slope polygon in general, and the étale locus in particular. We will describe this control in Corollary 3.4.7 below. To obtain this control, we prove a more refined result which we will use in the application to Rapoport-Zink spaces. This requires the use of some rings defined in terms of the seminorms $|\cdot|_{x,\rho}$, in order to perform base extension to "localize" around a point of $M(\overline{S}')$.

Definition 3.4.2. For T an open subset of $M(\overline{S}')$ and any $r \in (0, -\log \delta_0]$, let $\mathcal{R}_S^r(T)$ be the (separated) Fréchet completion of $\mathcal{R}_S^r$ with respect to the seminorms $\lambda(x, \rho)$ for all $x \in T$ and all ρ with $-\log \rho \in (0, r]$. Let $\mathcal{R}_S(T)$ be the union of the $\mathcal{R}_S^r(T)$ over all $r > 0$.

Remark 3.4.3. By Theorem 3.2.7, on one hand $\lambda(T) \subseteq \mu^{-1}(T)$ because $\mu \circ \lambda$ is the identity map. On the other hand, each $y \in \mu^{-1}(T)$ is dominated by $(\lambda \circ \mu)(y) \in \lambda(T)$. It follows that $\mathcal{R}_S(T)$ may also be characterized as the Fréchet completion of $\mathcal{R}_S^r$ with respect to the seminorms induced by the points in $\mu^{-1}(T)$. These form an open subset of $M(S) \times_K A_K[e^{-r}, 1)$.

Definition 3.4.4. We say a finite algebra B over $\mathcal{R}_S^r(T)$ is *admissible* if it is unramified with respect to $\lambda(x, \rho)$ for all $x \in T$ and all ρ with $-\log \rho \in (0, r]$. Such an algebra carries a supremum seminorm over $\lambda(x, \rho)$ for all $x \in T$ and all ρ with $-\log \rho \in (0, r]$ (computed as the supremum over all seminorms over $\lambda(x, \rho)$). We say an algebra C over $\mathcal{R}_S^r(T)$ is *pro-admissible* if it is the Fréchet completion of a union of finite admissible algebras over $\mathcal{R}_S^r(T)$ for the supremum seminorms.

Lemma 3.4.5. *Suppose we are given a point $x \in M(\overline{S}')$ and a finite étale algebra A over $\tilde{\mathcal{R}}_{\kappa_x}^{\mathrm{int}}$. Then there exist $r > 0$, an open neighborhood T of x in $M(\overline{S}')$, and an admissible finite algebra B over $\mathcal{R}_S^r(T)$ to which ϕ extends, such that $B \otimes_{\mathcal{R}_S^r(T)} \tilde{\mathcal{R}}_{\kappa_x}^{\mathrm{int}}$ contains A.*

Proof. Approximate sufficiently well the minimal polynomials of some generators of A over $\tilde{\mathcal{R}}_{\kappa_x}^{\mathrm{int}}$. □

Using the rings $\mathcal{R}_S^r(T)$, we can give a strong analogue for geometric families of Theorem 2.4.8. The proof follows Hartl's [20, Proposition 1.7.2], which in turn is based on [27, Lemma 6.1.1]. (The latter is also the model for the proof of Theorem 2.4.8.)

Theorem 3.4.6. *Suppose $x \in M(\overline{S}')$ belongs to the étale locus of $\mathcal{E}$. Then there exist $r > 0$, an open neighborhood T of x in $M(\overline{S}')$, and a pro-admissible algebra C over $\mathcal{R}_S(T)$ to which ϕ extends, such that $\mathcal{E} \otimes_{\mathcal{R}_S^r} C$ admits a basis of horizontal sections.*

Sketch of proof. Since $\mathcal{E}_x$ is étale, $\mathcal{E}_x$ is represented by a finite free module over $\tilde{\mathcal{R}}_{\kappa_x}^r$ for some $r \in (0, -\log \delta_0]$, admitting a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ on which ϕ acts via a matrix $A \in \mathrm{GL}_n(\tilde{\mathcal{R}}_{\kappa_x}^{r/q} \cap \tilde{\mathcal{R}}_{\kappa_x}^{\mathrm{int}})$. Let ℓ be an algebraic closure of κ_x. By Theorem 1.4.7 and the assumption that $\mathcal{E}_x$ is étale, there exists $U \in \mathrm{GL}_n(\tilde{\mathcal{R}}_\ell^r)$ such that $U^{-1} A \phi(U)$ is the identity matrix I_n.

By Lemma 3.4.5, we can choose a nonnegative integer m, an open neighborhood T of x in $M(\overline{S})$ an admissible finite algebra B over $\mathcal{R}_S^{r/q^m}(T)$ to which ϕ extends, and matrices $V, W \in M_n(\phi^{-m}(B))$, such that $\lambda(y, \rho)(VW - I_n) < 1$ for all $y \in T$ and all $\rho \in [e^{-r}, e^{-r/q}]$, and $\lambda(y, \rho)(V^{-1} A \phi(V) - I_n) < 1$ for all $y \in T$ and $\rho = e^{-r/q}$. By replacing r by r/q^m if necessary, we may force ourselves into the case $m = 0$.

By arguing now as in [27, Lemma 6.1.1, Lemma 6.2.1], we can construct $W \in \mathrm{GL}_n(B)$ with $\lambda(y, \rho)(W^{-1} A \phi(W) - I_n) < 1$ for all $y \in T$ and all $\rho \in [e^{-r}, 1)$. From here, it is straightforward to construct C. □

Corollary 3.4.7. *The étale locus of $\mathcal{E}$ is an open subset of $M(\overline{S}')$.*

It should be possible to prove a slightly stronger result using a slightly more complicated argument; since we do not need this here, we leave it as an open problem.

Conjecture 3.4.8. *The slope polygon of $\mathcal{E}_x$ is lower semicontinuous as a function on $M(\overline{S}')$.*

Remark 3.4.9. Note that Corollary 3.4.7 is included in Conjecture 3.4.8 because the possible values for the slope polygon are discrete. By contrast, in the arithmetic case, lower semicontinuity of the slope polygon would only imply that the étale locus is locally closed.

3.5 Relative p-adic Hodge theory

It is expected that the correspondence between Galois representations and (ϕ, Γ)-modules should extend to a correspondence between representations of arithmetic fundamental groups and geometric families of (ϕ, Γ)-modules. Using techniques introduced by Faltings, Andreatta and Brinon [1] have introduced such a correspondence under somewhat restrictive hypotheses. (See the work of Andreatta and Iovița [2] for some typical applications.)

Although the base ring $S = K\langle T_1, \dots, T_n, T_1^{-1}, \dots, T_n^{-1}\rangle$ which we will use later does satisfy the hypotheses of Andreatta and Brinon, what we wish to do still falls outside their framework because we do not have an *étale* object over that whole space. Our objects will only be étale over a rather peculiar subspace which is not itself the product of a relative annulus with a subspace of the base. Nonetheless, one half of the correspondence, the passage from geometric families of (ϕ, Γ)-modules to representations of fundamental groups, is sufficiently explicit that we can carry it out by hand. (By contrast, in the case of absolute families, it is the passage from Galois representations to (ϕ, Γ)-modules that works most easily.)

Hypothesis 3.5.1. Throughout § 3.5, take $\omega = p^{-p/(p-1)}$. Let K be a complete (but not necessarily finite) unramified extension of $\mathbb{Q}_p$. Put $S = K\langle T_1, \dots, T_m, T_1^{-1}, \dots, T_m^{-1}\rangle$ for some nonnegative integer m. Equip S with the absolute Frobenius lift ϕ_S taking T_i to T_i^p for $i = 1, \dots, m$. Fix the isomorphism $\mathbf{B}^\dagger_{K,\mathrm{rig}} \widehat{\otimes}_K S \cong \mathcal{R}_S$ taking $\pi \otimes 1$ to z. Equip $\mathcal{R}_S$ with the split absolute Frobenius lift induced by the usual Frobenius lift $\pi \mapsto (\pi + 1)^p - 1$ on $\mathbf{B}^\dagger_{K,\mathrm{rig}}$ and the Frobenius lift ϕ_S on S.

Definition 3.5.2. Choose a coherent sequence ζ_{p^n} of primitive p^n-th roots of unity. Let Φ_n be the minimal polynomial of ζ_{p^n} over K. Identify $M(W(\overline{S}^{\mathrm{perf}}[\frac{1}{p}]))$ with the inverse limit of the system $\cdots \xrightarrow{\phi} M(S) \xrightarrow{\phi} M(S)$. Given a sequence $x = (\dots, x_1, x_0)$ in this limit, let $y_n \in M(S((z)))$ be the unique extension of x_n for which $y_n(\Phi_n(\pi + 1)) = 0$. These again form a coherent sequence, so define an element y of $M(W(\overline{S}'^{,\mathrm{perf}})[\frac{1}{p}])$. Let ψ denote the map $x \mapsto y$.

Definition 3.5.3. Consider the semidirect product $\tilde{\Gamma}_K \cong \Gamma_K \ltimes \mathbb{Z}_p^m$ acting on $\mathcal{R}_S$ with Γ_K acting on $\mathbf{B}^\dagger_{K,\mathrm{rig}}$ as usual and acting trivially on S, and with $(e_1, \dots, e_m) \in \mathbb{Z}_p^m$ sending T_i to

$(1+\pi)^{e_i}T_i$. We define a *geometric family of* $(\phi, \tilde{\Gamma}_K)$*-modules* over $\mathcal{R}_S$ by analogy with the definition of geometric families of (ϕ, Γ_K)-modules.

Definition 3.5.4. Let $\mathcal{E}$ be a geometric family of $(\phi, \tilde{\Gamma}_K)$-modules over $\mathcal{R}_S$. Let $U \subseteq M(\overline{S}')$ denote the étale locus of $\mathcal{E}$.

Given any $x_0 \in M(S)$, lift x_0 to some $x \in M(W(\overline{S}^{\mathrm{perf}})[\frac{1}{p}])$. Suppose that $y = \psi(x)$ belongs to U. By Theorem 3.4.6, we obtain a real number $r > 0$, an open neighborhood T of $\mu(y)$ in $M(\overline{S}')$, and a pro-admissible finite algebra C over $\mathcal{R}_S(T)$ to which ϕ extends, such that $\mathcal{E} \otimes_{\mathcal{R}_S^r} \mathcal{R}_S^r(T)$ admits a basis of horizontal sections.

By Remark 3.4.3, for n sufficiently large, $\mathcal{R}_S^r(T)$ maps to the coordinate ring of an open neighborhood of y_n. By pulling back along ψ, we obtain an open neighborhood V_n of x_n in $M(S)$ and a profinite étale cover of $V_n \times_K K(\mu_{p^n})$ with automorphism group contained in $\mathrm{GL}_{\mathrm{rank}(\mathcal{E})}(\mathbb{Z}_p)$.

We may use the action of $\tilde{\Gamma}_K$ to perform Galois descent on the profinite étale cover; this yields a profinite étale cover of some open neighborhood V_0 of x_0, again with automorphism group contained in $\mathrm{GL}_{\mathrm{rank}\,\mathcal{E}}(\mathbb{Z}_p)$. We thus obtain from this data a $\mathbb{Z}_p$-local system on V_0 in the sense of de Jong [15].

If we compare the $\mathbb{Z}_p$-local systems on two overlapping open subsets of U, we only get a canonical isomorphism of the induced $\mathbb{Q}_p$-local systems. (That is because in Theorem 3.4.6, only the $\mathbb{Q}_p$-span of the horizontal sections is independent of the choice of the basis, not the $\mathbb{Z}_p$-span.) As a result, we produce a $\mathbb{Q}_p$-local system on all of U.

3.6 Rapoport-Zink spaces

We conclude with an application of geometric families of ϕ-modules to the study of Rapoport-Zink period domains. While the conjecture involves an arbitrary reductive Lie group, we restrict to the case of GL_n for ease of exposition. See [21, §1] for a brief introduction to the general case, and the original book of Rapoport and Zink [41] for further details.

Definition 3.6.1. Let K_0 be the completion of the maximal unramified extension of $\mathbb{Q}_p$. Fix a positive integer n and a multiset HT of n integers. Let $\mathcal{F}$ be the flag variety over K_0 parametrizing exhaustive filtrations with Hodge-Tate weights equal to HT; the trivial bundle $\mathcal{O}_{\mathcal{F}}^{\oplus n}$ carries a universal filtration $\mathrm{Fil}^{\cdot}$.

Fix a choice of $b \in \mathrm{GL}_n(K_0)$. For each point x of the Berkovich analytification $\mathcal{F}^{\mathrm{an}}$ of $\mathcal{F}$, we obtain a filtered ϕ-module D_x over $\mathcal{H}(x)$ with underlying space $\mathcal{H}(x)^n$, action of ϕ given by b, and filtration specified by the pullback of the universal filtration over $\mathcal{F}$.

Theorem 3.6.2 (Rapoport-Zink)**.** *There exists an open subspace $\mathcal{F}^{wa}$ of the Berkovich analytification $\mathcal{F}^{\mathrm{an}}$ of $\mathcal{F}$, such that for $x \in \mathcal{F}^{\mathrm{an}}$, $x \in \mathcal{F}^{wa}$ if and only if D_x is weakly admissible.*

Proof. See [41, Proposition 1.36]. □

At first glance, it might seem reasonable to construct a $\mathbb{Q}_p$-local system over $\mathcal{F}^{wa}$ whose restriction to any rigid analytic point x is the representation space of the crystalline Galois representation associated to D_x. It was observed by Rapoport and Zink that this is not possible; one must instead replace $\mathcal{F}^{wa}$ by some open subspace $\mathcal{F}^a$ which happens to contain the same rigid analytic points as $\mathcal{F}^{wa}$. Hartl defines a candidate for this space using a variant of the field of norms construction, as follows.

Definition 3.6.3. Given $x \in \mathcal{F}^{\mathrm{an}}$, let $\mathbb{C}_x$ be a completed algebraic closure of $\mathcal{H}(x)$. Let $\tilde{\mathbf{E}}^+$ be the set of sequences $(a_0, a_1, \dots)$ in $\mathfrak{o}_{\mathbb{C}_x}/p\mathfrak{o}_{\mathbb{C}_x}$ for which $a_n = a_{n+1}^p$ for each n; these naturally form a ring, and the function $v_{\tilde{\mathbf{E}}}$ carrying a nonzero sequence $(a_0, a_1, \dots)$ to the stable value of $p^n v_p(a_n)$ is a valuation on $\tilde{\mathbf{E}}^+$.

Let $\epsilon = (\epsilon_0, \epsilon_1, \dots) \in \tilde{\mathbf{E}}^+$ be an element with $\epsilon_0 = 1$ and $\epsilon_1 \neq 1$. It turns out that $\tilde{\mathbf{E}} = \operatorname{Frac} \tilde{\mathbf{E}}^+$ is complete and algebraically closed. In case $\mathcal{H}(x)$ is finite over $\mathbb{Q}_p$, the integral closure of $\mathbb{F}_p((\epsilon - 1))$ in $\tilde{\mathbf{E}}$ is dense in $\tilde{\mathbf{E}}$.

Define the element $t \in \mathbf{B}^\dagger_{K_0,\mathrm{rig}}$ by putting $\pi = [\epsilon] - 1$ and

$$t = \log[\epsilon] = \sum_{i=1}^\infty \frac{(-1)^{i-1}}{i} \pi^i.$$

Using this homomorphism θ, one can then imitate the passage from $M(D)$ to $M'(D)$ using the filtration to make a ϕ-module over $\tilde{\mathcal{R}}_{\mathbb{C}_x}$, and declare D to be *admissible* if the resulting ϕ-module is étale. Note that admissible implies weakly admissible by elementary properties of slope filtrations. Note also that for a rigid analytic point, weakly admissible implies admissible by Theorem 1.3.14, but this fails for general points; see for instance [22, Example 3.6].

The main theorem is then the following, which answers a question of Rapoport and Zink [41, p. 29]. In the important case where the Hodge-Tate weights belong to $\{0, 1\}$, part (a) is due to Hartl [21, Theorem 5.2] using a slightly different argument.

Theorem 3.6.4. *Let $\mathcal{F}^a$ be the subset of $x \in \mathcal{F}$ for which D_x is admissible.*

(a) The space $\mathcal{F}^a$ is an open subspace of $\mathcal{F}^{wa}$.

(b) There exists a $\mathbb{Q}_p$-local system on $\mathcal{F}^a$ whose restriction to any rigid analytic point x is the representation space of the crystalline Galois representation associated to D_x.

Proof. Recall that $\mathcal{F}$ is covered by Zariski open subsets which are isomorphic to affine spaces (the exact shape of these being unimportant here). We may thus find an affinoid subspace of $\mathcal{F}^{\mathrm{an}}$ containing x and isomorphic to $M(S)$ for $S = K_0\langle T_1, \dots, T_m, T_1^{-1}, \dots, T_m^{-1}\rangle$ for some m. (Note that we use polycircles rather than polydiscs; this is possible because the closed unit disc $|T| \leq 1$ can be covered by the two closed unit circles $|T| = 1$ and $|T - 1| = 1$, so a polydisc can be covered by finitely many polycircles.)

Equip $D = S^{\oplus n}$ with the universal filtration over $M(S)$. Equip $\mathbf{B}^\dagger_{K_0,\mathrm{rig}} \widehat{\otimes}_{K_0} S \cong \mathcal{R}_S$ with the actions of ϕ and $\tilde{\Gamma}_{K_0}$ suggested in Hypothesis 3.5.1. If we equip $M(S)$ with the ϕ-action specified by b and the trivial action of $\tilde{\Gamma}_{K_0}$, we obtain actions of ϕ and $\tilde{\Gamma}_{K_0}$ on $M(D) = \mathbf{B}^\dagger_{K_0,\mathrm{rig}} \widehat{\otimes}_{K_0} D$ which turn it into a geometric family of $(\phi, \tilde{\Gamma}_{K_0})$-modules. We can modify using the universal filtration over $\mathcal{F}$ to obtain another geometric family $M'(D)$ of $(\phi, \tilde{\Gamma}_{K_0})$-modules over $\mathcal{R}_S$. Using this family, we deduce (a) and (b) using the construction of Definition 3.5.4. □

Remark 3.6.5. The original case of interest is when all of the Hodge-Tate weights belong to $\{0, 1\}$. This case pertains to p-divisible groups in the following fashion. Let G be a p-divisible group (Barsotti-Tate group) of height h and dimension d. For any complete discrete valuation ring $\mathfrak{o}_K$ of characteristic 0 with residue field $\mathbb{F}_p^{\mathrm{alg}}$, and any deformation of G to a p-divisible group $\tilde{G}$ over $\mathfrak{o}_K$, Grothendieck and Messing [36] associate an extension

$$0 \to (\operatorname{Lie} \tilde{G}^\vee)_K^\vee \to \mathbb{D}(G)_K \to \operatorname{Lie} \tilde{G}_K \to 0$$

where $\mathbb{D}$ denotes the crystalline Dieudonné module functor. We thus end up with a K-point in the Grassmannian $\mathcal{F}$ of $(h-d)$-dimensional subspaces of $\mathbb{D}(G)_{K_0}$.

Grothendieck asked [19] which points of $\mathcal{F}$ can occur in this fashion. This question remains open; however, Rapoport and Zink proved [41, 5.16] that all such points belong to the image of a certain *period morphism* from the generic fibre of a certain universal deformation space. Using results of Faltings, Hartl [22, Theorem 3.5] has shown that his space $\mathcal{F}^a$ is exactly the image of the Rapoport-Zink period morphism.

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