Christol’s theorem and its analogue for generalized power series, part 1

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Challenges in Combinatorics on Words
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5. Proof of Christol’s theorem: algebraic implies automatic
6. Preview of part 2
Let $K$ be any field. The ring of formal power series over $K$, denoted $K[[t]]$, consists of formal infinite sums $\sum_{n=0}^{\infty} f_n t^n$ added term-by-term:

$$\sum_{n=0}^{\infty} f_n t^n + \sum_{n=0}^{\infty} g_n t^n = \sum_{n=0}^{\infty} (f_n + g_n) t^n$$

and multiplied by formal series multiplication (convolution):

$$\sum_{n=0}^{\infty} f_n t^n \times \sum_{n=0}^{\infty} g_n t^n = \sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} f_i g_{n-i} \right) t^n.$$
Formal Laurent series

A formal Laurent series over $K$ is a formal doubly infinite sum $\sum_{n \in \mathbb{Z}} f_n t^n$ with $f_n \in K$ such that only finitely many of the $f_n$ for $n < 0$ are nonzero. These again form a ring:

\[
\sum_{n \in \mathbb{Z}} f_n t^n + \sum_{n \in \mathbb{Z}} g_n t^n = \sum_{n \in \mathbb{Z}} (f_n + g_n) t^n
\]

\[
\sum_{n \in \mathbb{Z}} f_n t^n \times \sum_{n \in \mathbb{Z}} g_n t^n = \sum_{n \in \mathbb{Z}} \left( \sum_{i+j=n} f_i g_j \right) t^n.
\]

In fact these form a field, denoted $K((t))$. It is the fraction field of $K[[t]]$. 
Polynomials and power series

There is an obvious inclusion of the polynomial ring $K[t]$ into the formal power series ring $K[[t]]$. Since $K((t))$ is a field, this extends to an inclusion of the rational function field $K(t)$ into the formal Laurent series field $K((t))$.

**Proposition (easy)**

The image of $K(t)$ in $K((t))$ consists of those formal Laurent series $\sum_{n \in \mathbb{Z}} f_n t^n$ for which the sequence $f_0, f_1, \ldots$ satisfies a linear recurrence relation. That is, for some nonnegative integer $m$ there exist $c_0, \ldots, c_m \in K$ not all zero such that

$$c_0 f_n + \cdots + c_m f_{n+m} = 0 \quad (n = 0, 1, \ldots).$$
Let $K \subseteq L$ be an inclusion of fields. An element $x \in L$ is *algebraic* over $K$ (or *integral* over $K$) if there exists a monic polynomial $P[z] \in K[z]$ such that $P(x) = 0$. For example, $\sqrt{-1} \in \mathbb{C}$ is algebraic over $\mathbb{Q}$.

**Proposition**

*The set of $x \in L$ which are algebraic over $K$ is a subfield of $L$.***

**Proof.**

$x \in L$ is algebraic over $K$ if and only if all powers of $x$ lie in a finite-dimensional $K$-subspace of $L$. (We’ll see the proof later.)
Let us specialize to the inclusion \( K(t) \subset K((t)) \).

**Question**

*Can one give an explicit description of those elements of \( K((t)) \) which are algebraic over \( K(t) \), analogous to the description of \( K(t) \) in terms of coefficients?*

Amazingly, when \( K \) is a finite field this question has an affirmative answer in terms of combinatorics on words!
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Regular languages

Fix a finite set $\Sigma$ as the \textit{alphabet}. Let $\Sigma^*$ denote the set of finite words on $\Sigma$. A \textit{language} on $\Sigma$ is a subset $L$ of $\Sigma^*$. We write $xy$ for the concatenation of the words $x$ and $y$.

A \textit{deterministic finite automaton} $\Delta$ on $\Sigma$ consists of a finite state set $S$, an \textit{initial state} $s_0 \in S$, and a \textit{transition function} $\delta : S \times \Sigma \rightarrow S$. The automaton induces a function $g_\Delta : \Sigma^* \rightarrow S$ by

$$g_\Delta(\emptyset) = s_0, \quad g_\Delta(x\sigma) = \delta(g_\Delta(x), \sigma).$$

Any language of the form $g_\Delta^{-1}(S_1)$ for some $S_1 \subseteq S$ is \textit{accepted} by $\Delta$.

Any language accepted by some automaton is said to be \textit{regular}. It is equivalent to ask that the language be accepted by some regular expression or by some nondeterministic finite automaton. In particular, reversing all strings in a regular language yields a regular language.
Regular languages

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More on regular languages

Let \( L \) be a language on \( \Sigma \). Define an equivalence relation on \( \Sigma^* \) by declaring that \( x \sim_L y \) if and only if for all \( z \in \Sigma^* \), \( xz \in L \) if and only if \( yz \in L \).

**Theorem (Myhill-Nerode)**

The language \( L \) is regular if and only if \( \Sigma^* \) splits into finitely many equivalence classes under \( \sim_L \).

**Sketch of proof.**

If \( L \) is accepted by a finite automaton, then any two words leading to the same state are equivalent. Conversely, if there are finitely many equivalence classes, these correspond to the states of a minimal finite automaton which accepts \( L \).
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Regular functions

Let $U$ be a finite set. Let $f : \Sigma^* \rightarrow U$ be a function. Define another equivalence relation on $\Sigma^*$ by declaring that $x \sim_f y$ if and only if for all $z \in \Sigma^*$, $f(xz) = f(yz)$.

We say that $f$ is regular if $f^{-1}(u)$ is a regular language for all $u \in U$. Equivalently, there exist an automaton $\Delta = (S, s_0, \delta)$ and a function $h : S \rightarrow U$ such that $f = h \circ g_\Delta$ (in which case we say that $\Delta$ accepts $f$).

**Theorem (Myhill-Nerode for functions)**

The function $f$ is regular if and only if $\Sigma^*$ splits into finitely many equivalence classes under $\sim_f$.

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Finite fields

For the remainder of these two talks, fix a prime number $p > 0$ and let $q$ be a power of $p$. Up to isomorphism, there is a unique finite field of $q$ elements, which we denote by $\mathbb{F}_q$. (This object is not unique up to unique isomorphism, but never mind.)

Every finite extension of $\mathbb{F}_q$ is again a finite field, and thus isomorphic to $\mathbb{F}_{q'}$ where $q'$ must be a power of $q$. Conversely, every power of $q$ as the cardinality of a finite extension of $\mathbb{F}_q$.

For example, we can write

$$\mathbb{F}_4 \cong (\mathbb{Z}/2\mathbb{Z})[z]/(z^2 + z + 1)$$
$$\mathbb{F}_9 \cong (\mathbb{Z}/3\mathbb{Z})[z]/(z^2 + 1).$$
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Frobenius

Since $\mathbb{F}_q$ is of characteristic $p$, the Frobenius map $x \mapsto x^p$ is a ring homomorphism. It is also injective, so it is in fact a field automorphism.

We will use frequently the fact that the $p$-th power map also induces a Frobenius endomorphism on $\mathbb{F}_q(t)$ and $\mathbb{F}_q((t))$. These maps are injective but not surjective: an element of $\mathbb{F}_q(t)$ (resp. $\mathbb{F}_q((t))$) is a $p$-th power if and only if it is a rational function (resp. Laurent series) in $t^p$. 
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Fix the alphabet $\Sigma = \{0, \ldots, p - 1\}$. We may identify nonnegative integers with words on $\Sigma$ using base-$p$ expansions. We will allow arbitrary leading zeroes.

For $f = \sum_{n \in \mathbb{Z}} f_n t^n \in \mathbb{F}_q((t))$, we identify $f$ with a function $f : \Sigma^* \rightarrow \mathbb{F}_q$ taking a base-$p$ expansion of $n$ (with any number of leading zeroes) to $f_n$. We say $f \in \mathbb{F}_q((t))$ is *automatic* if the corresponding function $f : \Sigma^* \rightarrow \mathbb{F}_q$ is regular.

**Theorem (Christol, 1979; Christol–Kamae–Mendès France–aRauzy, 1980)**

A formal Laurent series is algebraic over $\mathbb{F}_q(t)$ if and only if it is automatic.
The theorem of Christol

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Theorem (Christol, 1979; Christol–Kamae–Mendès France–aRauzy, 1980)

A formal Laurent series is algebraic over $\mathbb{F}_q(t)$ if and only if it is automatic.
Example: the Thue-Morse sequence

Take \( f = \sum_{n=0}^{\infty} f_n t^n \in \mathbb{F}_2((t)) \) with

\[
f_n = \begin{cases} 
1 & \text{if the number of 1's in the base-2 expansion of } n \text{ is even} \\
0 & \text{otherwise.}
\end{cases}
\]

Then \( f \) is automatic, e.g., for the regular expression

\[0^*(10^*10^*)^*\]

or the DFA

![DFA diagram](image)

and \( f \) is algebraic:

\[(1 + t)^3 f^2 + (1 + t)^2 f + t = 0.\]
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\begin{array}{c}
\text{start} \\
0 \\
1 \\
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0 \\
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Example: from the Putnam competition

Problem (1989 Putnam competition, problem A6)

Let $\alpha = 1 + a_1 x + a_2 x^2 + \cdots$ be a formal power series with coefficients in the field of two elements. Let

$$a_n = \begin{cases} 
1 & \text{if every block of zeros in the binary expansion of } n \text{ has an even number of zeros in the block} \\
0 & \text{otherwise.}
\end{cases}$$

Prove that $\alpha^3 + x\alpha + 1 = 0$. 

Application: the Hadamard product

For $f = \sum_{n \in \mathbb{Z}} f_n t^n$, $g = \sum_{n \in \mathbb{Z}} g_n t^n \in \mathbb{F}_q((t))$, define the Hadamard product

$$f \circ g = \sum_{n \in \mathbb{Z}} f_n g_n t^n.$$ 

Theorem (Furstenberg, 1967)

If $f, g \in \mathbb{F}_q((t))$ are algebraic over $\mathbb{F}_q(t)$, then so is $f \circ g$.

Sketch of proof.

Check the analogous assertion for automatic sequences, which is easy. See Allouche–Shallit, Theorem 12.2.6.

Note that $\mathbb{F}_q$ is special: over $\mathbb{Q}(t)$, $f$ is algebraic but not $f \circ f$ for

$$f = (1 - 4t)^{-1/2} = \sum_{n=0}^{\infty} \binom{2n}{n} t^n.$$
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Theorem (Furstenberg, 1967 for $f \in \mathbb{F}_q(t, u)$; Deligne, 1984)

Let $f = \sum_{m,n=0}^{\infty} f_{mn} t^m u^n$ be a bivariate formal power series over $\mathbb{F}_q$ which is algebraic over $\mathbb{F}_q(t, u)$. Then the diagonal series $\sum_{n=0}^{\infty} f_{nn} t^n$ is algebraic over $\mathbb{F}_q(t)$.

Proof.

This follows from a multivariate analogue of Christol’s theorem. See Allouche–Shallit, Theorem 14.4.2.

Conversely, every power series algebraic over $\mathbb{F}_q(t)$ arises as the diagonal of some $f \in \mathbb{F}_q(t, u)$ (Furstenberg, 1967). See Allouche–Shallit, Theorem 12.7.3.
Application: diagonals

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This follows from a multivariate analogue of Christol’s theorem. See Allouche–Shallit, Theorem 14.4.2.

Conversely, every power series algebraic over $\mathbb{F}_q(t)$ arises as the diagonal of some $f \in \mathbb{F}_q(t, u)$ (Furstenberg, 1967). See Allouche- Shallit, Theorem 12.7.3.
Application: transcendence results

The existence of Christol’s theorem makes it possible to prove much better transcendence results over $\mathbb{F}_q(t)$ than over $\mathbb{Q}$.

Theorem (Wade, 1941; Allouche, 1990 using Christol)

The “Carlitz $\pi$”

$$\pi_q = \prod_{k=1}^{\infty} \left( 1 - \frac{tq^k - t}{tq^{k+1} - t} \right)$$

is transcendental over $\mathbb{F}_q(t)$.

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6 Preview of part 2
Algebraicity in characteristic $p$

Recall that $f \in \mathbb{F}_q((t))$ is algebraic over $\mathbb{F}_q(t)$ if and only if the powers of $f$ all lie in a finite dimensional $\mathbb{F}_q(t)$-subspace of $\mathbb{F}_q((t))$. The following variant (with the same proof) will be useful.

**Proposition (Ore)**

The element $f \in \mathbb{F}_q((t))$ is algebraic over $\mathbb{F}_q(t)$ if and only if $f, f^p, f^{p^2}, \ldots$ all belong to a finite-dimensional $\mathbb{F}_q(t)$-subspace of $\mathbb{F}_q((t))$.

**Proof.**

If $f$ is a root of a monic polynomial $P$ of degree $d$ over $\mathbb{F}_q(t)$, then every power of $f$ belongs to the $\mathbb{F}_q(t)$-linear span of $1, f, \ldots, f^{d-1}$. Conversely, if the inclusion holds, then any linear dependence among $f, f^p, f^{p^2}, \ldots$ gives rise to a polynomial over $\mathbb{F}_q(t)$ having $f$ as a root.
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Let \( f = \sum_{n \in \mathbb{Z}} f_n t^n \in \mathbb{F}_q((t)) \) be automatic. Choose an automaton \( \Delta = (S, s_0, \delta) \) and a function \( h : S \to \mathbb{F}_q \) such that \( f = h \circ g_{\Delta} \). Define

\[
e_s = \sum_{n \geq 0, g_{\Delta}(n) = s} t^n \quad (s \in S).
\]

Note that

\[
f = \sum_{s \in S} h(s) e_s,
\]

so it suffices to check that the \( e_s \) are algebraic. The key relation is

\[
e_s = \sum_{s' \in S, i \in \{0, \ldots, p-1\} : \delta(s', i) = s} e_{s'}^p t^i.
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(This is correct even for \( s = s_0 \) because we must have \( \delta(s_0, 0) = s_0 \).)
Automatic implies algebraic

Let \( f = \sum_{n \in \mathbb{Z}} f_n t^n \in \mathbb{F}_q((t)) \) be automatic. Choose an automaton \( \Delta = (S, s_0, \delta) \) and a function \( h : S \to \mathbb{F}_q \) such that \( f = h \circ g_\Delta \). Define

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Since we are in characteristic $p$, the $p$-th power map is an automorphism. Hence for each $m \geq 0$,

$$e_s^{p^m} = \sum_{s', i : \delta(s', i) = s} e_s^{p^{m+1}} t^{ip^m}.$$ 

Therefore $e_s^{p^m}$ is contained in the $\mathbb{F}_q(t)$-span of the $e_s^{p^{m+1}}$.

By induction, $\{e_s^{p^i} : s \in S, i = 0, \ldots, m\}$ is contained in the $\mathbb{F}_q(t)$-span of $\{e_s^{p^m} : s \in S\}$. In particular, $e_s, e_s^p, \ldots, e_s^{p^m}$ belong to an $\mathbb{F}_q(t)$-vector space whose dimension is bounded independent of $m$. It follows that $e_s$ is algebraic, as then is $f$. 
Since we are in characteristic $p$, the $p$-th power map is an automorphism. Hence for each $m \geq 0$,

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Proof of Christol’s theorem: algebraic implies automatic

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Decimation of power series

The proof in this direction uses a criterion for automaticity analogous to that of algebraicity, except with the $p$-th power map replaced by some maps in the opposite direction.

Lemma

For $f \in \mathbb{F}_q((t))$, there is a unique way to write

$$f = d_0(f)^p + td_1(f)^p + \cdots + t^{p-1}d_{p-1}(f)^p$$

with $d_0(f), \ldots, d_{p-1}(f) \in \mathbb{F}_q((t))$.

Proof.

Sort the terms of $f$ by their degree modulo $p$, then recall that an element of $\mathbb{F}_q((t))$ is a power series in $t^p$ if and only if it is a $p$-th power. \qed
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Decimation and automaticity

We view $d_0, \ldots, d_{p-1}$ as maps from $\mathbb{F}_q((t))$ to itself. These maps are additive:

$$d_i(f + g) = d_i(f) + d_i(g) \quad (f, g \in \mathbb{F}_q((t))).$$

but not multiplicative per se. Something similar is true, though:

$$d_i(f^p g) = f d_i(g) \quad (f, g \in \mathbb{F}_q((t))).$$

Using the $d_i$, we can give a finiteness criterion for automaticity.

**Proposition**

For $f \in \mathbb{F}_q((t))$, $f$ is automatic if and only if $f$ is contained in a finite subset of $\mathbb{F}_q((t))$ closed under $d_i$ for $i = 0, \ldots, p-1$.

**Proof.**

This is a reformulation of Myhill-Nerode.
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Decimation of rational functions

We define the \textit{degree} of a nonzero rational function $f \in \mathbb{F}_q(t)$ by writing $f = g/h$ with $g, h \in \mathbb{F}_q[t]$ nonzero and coprime, then putting

$$\deg(f) = \max\{\deg(g), \deg(h)\}.$$ 

By convention, $\deg(0) = -\infty$.

\textbf{Lemma}

\textit{For} $f \in \mathbb{F}_q(t)$ \textit{and} $i = 0, \ldots, p - 1$, \textit{we have} $d_i(f) \in \mathbb{F}_q(t)$ \textit{and} $\deg(d_i(f)) \leq \deg(f)$.

\textbf{Proof.}

We have

$$d_i(f) = d_i(gh^{p-1}/h^p) = d_i(gh^{p-1})/h$$

and $\deg(d_i(gh^{p-1})) \leq \deg(gh^{p-1})/p \leq \deg(f)$.\hfill\(\square\)
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We define the degree of a nonzero rational function \( f \in \mathbb{F}_q(t) \) by writing \( f = g/h \) with \( g, h \in \mathbb{F}_q[t] \) nonzero and coprime, then putting

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More on algebraicity in characteristic \( p \)

**Proposition (Ore)**

*If \( f \in \mathbb{F}_q((t)) \) is algebraic over \( \mathbb{F}_q(t) \), then \( f \) is in the \( \mathbb{F}_q(t) \)-span of \( f^p, f^{p^2}, \ldots \).*

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We have a relation

\[
f^{p^l} = h_1 f^{p^{l+1}} + \cdots + h_m f^{p^{l+m}}
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for some \( l, m \geq 0 \) and \( h_1, \ldots, h_m \in \mathbb{F}_q(t) \). If \( l > 0 \), then also

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f^{p^{l-1}} = d_0(h_1)f^{p^l} + \cdots + d_0(h_m)f^{p^{l+m-1}},
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so we may force \( l = 0 \). \( \square \)
Proof of Christol's theorem: algebraic implies automatic

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Suppose that $f \in \mathbb{F}_q((t))$ is algebraic. We then have

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for some $h_1, \ldots, h_m \in \mathbb{F}_q(t)$. Put $H = \max_j \{ \deg(h_j) \}$ and

$$G = \{ g \in \mathbb{F}_q((t)) : g = \sum_{j=0}^m e_j f^{p^j}, e_j \in \mathbb{F}_q(t), \deg(e_j) \leq H \}.$$

Each $e_j$ is limited to a finite set, so $G$ is finite. But for $g \in G$ and $i = 0, \ldots, p - 1$,

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In part 2, we will see how to replace the field $\mathbb{F}_q((t))$ by a field of generalized power series so that the analogue of Christol’s theorem holds and does determine a full algebraic closure of $\mathbb{F}_q(t)$. 
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