The Fourier transform on the affine line
and the Weil Conjectures

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Let $X$ be a smooth proper variety over the finite field $\mathbb{F}_q$. The zeta function of $X$ is

$$Z_X(T) = \prod_{x \in X} (1 - T^{\deg(x)})^{-1}$$

$$= \exp \left( \sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} \right).$$

Weil Conjectures (imprecise form): $Z_X(T) \in \mathbb{Z}[T]$ represents a rational function of $T$ with zeroes and poles only on the circles $|T| = q^{-i/2}$ for $i = 0, \ldots, 2 \dim(X)$.

WC proved using étale ($\ell$-adic) cohomology: Grothendieck et al, Deligne. Here we analogize a variation due to Laumon.
The WC and de Rham cohomology

Dwork’s original proof of the rationality of $Z_X(T)$ suggests building a de Rham cohomology for varieties over $\mathbb{F}_q$ using lifts to characteristic 0. The coefficients in this theory are Berthelot’s “arithmetic $D^\dagger$-modules”; here $D^\dagger$ is a certain $p$-adic completion of a ring of differential operators.

This method has the advantage of being useful for explicit computations; see Denef’s talk.
The affine line

Let $K$ be a finite extension of $\mathbb{Q}_p(\zeta_p)$ with residue field $\mathbb{F}_q$. Then $K$ contains $\pi$ such that $\pi^{p-1} = -p$.

In Berthelot’s theory, the “ring of functions on $\mathbb{A}^1_{\mathbb{F}_q}$” $\mathcal{O}_{\mathbb{A}^1}$ consists of sums

$$\sum_{n=0}^{\infty} c_n t^n \quad (c_n \in K)$$

for which there exists $\rho > 1$ with

$$\lim_{n \to \infty} |c_n| \rho^n = 0.$$ 

The ring $\mathcal{D}^{\dagger}_{\mathbb{A}^1}$ of differential operators consists of double sums

$$\sum_{m,n=0}^{\infty} c_{m,n} t^n \partial^n \quad (c_{m,n} \in K)$$

for which there exists $\rho > 1$ with

$$\lim_{n \to \infty} |c_{m,n}| \rho^{m+n} = 0.$$ 

Here $\partial$ acts as $\pi^{-1} \frac{d}{dt}$. 
The Fourier transform on the affine line

We will mostly be interested in coherent (left) $\mathcal{D}^\dagger$-modules; on these, one has a Fourier transform given by pullback along

$$t \mapsto \partial, \partial \mapsto -t.$$  

Key point: this is anti-involutive, so preserves irreducibility.

Better, one should restrict to “holonomic” $\mathcal{D}^\dagger$-modules; however, the characterization of these is somewhat awkward.
Frobenius actions

An overconvergent $F$-isocrystal on $\mathbb{A}^1_{\mathbb{F}_q}$ is a finite free $\mathcal{O}_{\mathbb{A}^1}$-module $M$ equipped with a connection $\nabla : M \to M \otimes \Omega^1$ (were $\Omega^1 = \mathcal{O}_{\mathbb{A}^1} dt$ and $d : \mathcal{O}_{\mathbb{A}^1} \to \Omega^1$ is formal derivation), plus an isomorphism $F : \sigma^* M \to M$ of modules with connection, for

$$\sigma : \sum c_n t^n \mapsto \sum c_n t^{qn}.$$ 

Such an object is automatically a $\mathcal{D}^{\dagger}$-module (nontrivial calculation). In fact, a $\mathcal{D}^{\dagger}$-module with Frobenius action is an overconvergent $F$-isocrystal if and only if it is coherent over $\mathcal{O}_{\mathbb{A}^1}$.

Custom: view $F$ as a $\sigma$-linear map on $M$ itself (so $F(r v) = r^\sigma F(v)$).
Frobenius actions (continued)

For \( x \in \overline{\mathbb{F}}_q \) (corresponding to a geometric point of \( \mathbb{A}^1_{\overline{\mathbb{F}}_q} \)), \([x] \) the Teichmüller lift, and \( K_x = K([x]) \subset K^{unr} \), put

\[
M_x = (M \otimes_K K_x)/(t - [x])(M \otimes_K K_x).
\]

Then \( F^{\text{deg}(x)} \) acts linearly on \( M_x \), which is a vector space over \( K_x \) of dimension \( \text{rank}(M) \).

Also, \( F \) acts linearly on

\[
H^1(M) = \text{coker}(\nabla : M \to M \otimes \Omega^1),
\]

which is finite dimensional over \( K \) (by “Crew’s conjecture” on \( p \)-adic differential equations: André, Mebkhout, K).

Analogue in étale cohomology: a lisse sheaf. Holonomic \( \mathcal{D}^\dagger \)-modules would correspond to constructible sheaves.
Cohomological interpretation

For $f \in \mathcal{O}_{\mathbb{A}^1}$, let $L_f$ be the rank 1 $\mathcal{O}_{\mathbb{A}^1}$-module with Frobenius and connection given by

$$\nabla v = \pi v \otimes df, \quad F v = \exp(\pi f^\sigma - \pi f);$$

note that $L_f^\otimes p$ is trivial.

If $M$ is an overconvergent $F$-isocrystal and its Fourier transform $\hat{M}$ turns out to be one too, then $\hat{M}_x$ can be identified with $H^1(M \otimes L_{xt})$.

In particular, for any given $M$,

$$\hat{M} \otimes L_{P(t)}$$

is an overconvergent $F$-isocrystal if $\deg(P) = N$ with $\gcd(N, p) = 1$ and $N$ large. (Key point: $\dim H^1(M \otimes L_{P(t)})$ depends only on $N$ when $N$ is large, by a version of the Grothendieck-Ogg-Shafarevich formula.)
Deligne’s “Weil II” for $\mathcal{D}^\dagger$-modules

Theorem: let $M$ be an overconvergent $F$-isocrystal on $\mathbb{A}^1_{\overline{\mathbb{F}}_q}$. Suppose $M$ is pure of weight $i$: for each $x \in \overline{\mathbb{F}}_q$, $F^{\deg(x)}$ acts on $M_x$ via a linear transformation whose eigenvalues are algebraic numbers of complex norm $q^{i \cdot \deg(x)/2}$. Then $F$ acts on $H^1(M)$ via a linear transformation whose eigenvalues are algebraic numbers, each of complex norm $q^{(i+1-j) \cdot \deg(x)/2}$ for some nonnegative integer $j$.

This implies the Weil Conjectures using the formalism of rigid cohomology, essentially as for étale cohomology. (Finiteness of $H^1(M)$ relies on “Crew's conjecture” on $p$-adic differential equations, now a theorem of André, Mebkhout, K.)
To prove “Weil II”, one first shows that $M$ is irreducible and “real” (the characteristic polynomial of $F^{\deg(x)}$ on $M_x$ has coefficients which are totally real algebraic numbers), then $M$ is pure of some slope.

One then argues that if $M$ is irreducible, real, and pure of weight $i$, and $\hat{M}$ is an overconvergent $F$-isocrystal, then $\hat{M}$ is also irreducible and real. Hence $\hat{M}$ is pure of some weight, which is forced to be $i + 1$ by Poincaré duality. (Note: two Fourier transforms leave the Fourier transform twisted by $q$.) The stalk of the Fourier transform at 0 is precisely $H^1(M)$. 

$p$-adic “Weil II” after Laumon
In general, one can fit $M$ into a family where the previous paragraph applies, by twisting by $L_{at}^N$ for $N$ large. View these as a family over the $\alpha$-line, then use degeneration techniques ("local monodromy") to prove the desired result. (Roughly, $H^1(M \otimes L_{at}^N)$ forms an over-convergent $F$-isocrystal on the affine line minus the origin, whose "local monodromy around zero" contains $H^1(M)$.)