The Fourier transform on the affine line and the Weil Conjectures

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The Weil Conjectures

Let X be a smooth proper variety over the finite field $\mathbb{F}_q$. The zeta function of X is

$$\begin{aligned} Z_X(T) &= \prod_{x \in X} (1 - T^{\deg(x)})^{-1} \\ &= \exp\left(\sum_{n=1}^{\infty} \frac{\# X(\mathbb{F}_{q^n})}{n} \right). \end{aligned}$$

Weil Conjectures (imprecise form): $Z_X(T) \in \mathbb{Z}[\![T]\!]$ represents a rational function of T with zeroes and poles only on the circles $|T| = q^{-i/2}$ for $i = 0, \ldots, 2\dim(X)$.

WC proved using étale (ℓ-adic) cohomology: Grothendieck et al, Deligne. Here we analogize a variation due to Laumon.

The WC and de Rham cohomology

Dwork's original proof of the rationality of $Z_X(T)$ suggests building a de Rham cohomology for varieties over $\mathbb{F}_q$ using lifts to characteristic 0. The coefficients in this theory are Berthelot's "arithmetic $\mathcal{D}^\dagger$-modules"; here $\mathcal{D}^\dagger$ is a certain p-adic completion of a ring of differential operators.

This method has the advantage of being useful for explicit computations; see Denef's talk.

The affine line

Let K be a finite extension of $\mathbb{Q}_p(\zeta_p)$ with residue field $\mathbb{F}_q$. Then K contains π such that $\pi^{p-1} = -p$.

In Berthelot's theory, the "ring of functions on $\mathbb{A}^1_{\mathbb{F}_q}$" $\mathcal{O}_{\mathbb{A}^1}$ consists of sums

$$\sum_{n=0}^{\infty} c_n t^n \qquad (c_n \in K)$$

for which there exists $\rho > 1$ with

$$\lim_{n\to\infty} |c_n|\rho^n = 0.$$

The ring $\mathcal{D}^{\dagger}_{\mathbb{A}^1}$ of differential operators consists of double sums

$$\sum_{m,n=0}^{\infty} c_{m,n} t^n \partial^n \qquad (c_{m,n} \in K)$$

for which there exists $\rho > 1$ with

$$\lim_{n\to\infty} |c_{m,n}|\rho^{m+n} = 0.$$

Here ∂ acts as $\pi^{-1}\frac{d}{dt}$.

The Fourier transform on the affine line

We will mostly be interested in coherent (left) $\mathcal{D}^\dagger$-modules; on these, one has a Fourier transform given by pullback along

$$t \mapsto \partial, \partial \to -t.$$

Key point: this is anti-involutive, so preserves irreducibility.

Better, one should restrict to "holonomic" $\mathcal{D}^\dagger$-modules; however, the characterization of these is somewhat awkward.

Frobenius actions

An *overconvergent F-isocrystal* on $\mathbb{A}^1_{\mathbb{F}_q}$ is a finite free $\mathcal{O}_{\mathbb{A}^1}$-module M equipped with a connection $\nabla : M \to M \otimes \Omega^1$ (were $\Omega^1 = \mathcal{O}_{\mathbb{A}^1}\, dt$ and $d : \mathcal{O}_{\mathbb{A}^1} \to \Omega^1$ is formal derivation), plus an isomorphism $F : \sigma^* M \to M$ of modules with connection, for

$$\sigma : \sum c_n t^n \mapsto \sum c_n t^{qn}.$$

Such an object is automatically a $\mathcal{D}^\dagger$-module (nontrivial calculation). In fact, a $\mathcal{D}^\dagger$-module with Frobenius action is an overconvergent F-isocrystal if and only if it is coherent over $\mathcal{O}_{\mathbb{A}^1}$.

Custom: view F as a σ-linear map on M itself (so $F(r\mathbf{v}) = r^\sigma F(\mathbf{v})$).

Frobenius actions (continued)

For $x \in \overline{\mathbb{F}_q}$ (corresponding to a geometric point of $\mathbb{A}^1_{\mathbb{F}_q}$), $[x]$ the Teichmüller lift, and $K_x = K([x]) \subset K^{\mathrm{unr}}$, put

$$M_x = (M \otimes_K K_x)/(t - [x])(M \otimes_K K_x).$$

Then $F^{\deg(x)}$ acts linearly on M_x, which is a vector space over K_x of dimension $\mathrm{rank}(M)$.

Also, F acts linearly on

$$H^1(M) = \mathrm{coker}(\nabla : M \to M \otimes \Omega^1),$$

which is finite dimensional over K (by "Crew's conjecture" on p-adic differential equations: André, Mebkhout, K).

Analogue in étale cohomology: a lisse sheaf. Holonomic $\mathcal{D}^\dagger$-modules would correspond to constructible sheaves.

Cohomological interpretation

For $f \in \mathcal{O}_{\mathbb{A}^1}$, let L_f be the rank 1 $\mathcal{O}_{\mathbb{A}^1}$-module with Frobenius and connection given by

$$\nabla \mathbf{v} = \pi \mathbf{v} \otimes df, \qquad F\mathbf{v} = \exp(\pi f^\sigma - \pi f);$$

note that $L_f^{\otimes p}$ is trivial.

If M is an overconvergent F-isocrystal and its Fourier transform $\widehat{M}$ turns out to be one too, then $\widehat{M}_x$ can be identified with $H^1(M \otimes L_{xt})$.

In particular, for any given M,

$$M \widehat{\otimes L_{P(t)}}$$

is an overconvergent F-isocrystal if $\deg(P) = N$ with $\gcd(N, p) = 1$ and N large. (Key point: $\dim H^1(M \otimes L_{P(t)})$ depends only on N when N is large, by a version of the Grothendieck-Ogg-Shafarevich formula.)

Deligne's "Weil II" for $\mathcal{D}^\dagger$-modules

Theorem: let M be an overconvergent F-isocrystal on $\mathbb{A}^1_{\mathbb{F}_q}$. Suppose M is *pure of weight* i: for each $x \in \overline{\mathbb{F}_q}$, $F^{\deg(x)}$ acts on M_x via a linear transformation whose eigenvalues are algebraic numbers of complex norm $q^{i \deg(x)/2}$. Then F acts on $H^1(M)$ via a linear transformation whose eigenvalues are algebraic numbers, each of complex norm $q^{(i+1-j)\deg(x)/2}$ for some nonnegative integer j.

This implies the Weil Conjectures using the formalism of rigid cohomology, essentially as for étale cohomology. (Finiteness of $H^1(M)$ relies on "Crew's conjecture" on p-adic differential equations, now a theorem of André, Mebkhout, K.)

p-adic "Weil II" after Laumon

To prove "Weil II", one first shows that M is irreducible and "real" (the characteristic polynomial of $F^{\deg(x)}$ on M_x has coefficients which are totally real algebraic numbers), then M is pure of some slope.

One then argues that if M is irreducible, real, and pure of weight i, and $\widehat{M}$ is an overconvergent F-isocrystal, then $\widehat{M}$ is also irreducible and real. Hence $\widehat{M}$ is pure of some weight, which is forced to be $i+1$ by Poincaré duality. (Note: two Fourier transforms leave the Fourier transform twisted by q.) The stalk of the Fourier transform at 0 is precisely $H^1(M)$.

p-adic "Weil II" after Laumon (contd.)

In general, one can fit M into a family where the previous paragraph applies, by twisting by L_{at^N} for N large. View these as a family over the a-line, then use degeneration techniques ("local monodromy") to prove the desired result. (Roughly, $H^1(M \otimes L_{at^N})$ forms an overconvergent F-isocrystal on the affine line minus the origin, whose "local monodromy around zero" contains $H^1(M)$.)